Uniform-Price Auctions with a Last Accepted Bid Pricing Rule

PRELIMINARY DRAFT

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Abstract

We model multi-unit auctions in which bidders' valuations are multidimensional private information. We show that the *last-accepted-bid* uniform-pricing rule admits a unique equilibrium with a simple characterization, while the *first-rejected-bid* uniform-pricing rule admits many equilibria, many of which provide zero expected revenue. In a natural example, equilibrium strategies in the last-accepted-bid auction are constructed from familiar strategies for single-unit first-price auctions. In contrast with the information pooling we prove to arise in the first-rejected-bid and *pay-as-bid* auctions, the unique equilibrium of the last-accepted-bid auction is fully revealing.

1 Introduction

In a uniform-price auction bidders submit demand functions to a seller who awards m units of a homogeneous good to the highest m bids at a single clearing price that applies to all bids.[1] These rules govern well-known large-scale auctions, such as those run by the U.S. Treasury and the independent system operators in charge of electricity distribution, but they have also been used to model decentralized oligopoly markets under the guise of

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[1] These rules are easily modified to accommodate bidders submitting supply curves to sell goods, sale of a divisible good and/or a seller using a non-constant supply (or demand) schedule. With indivisible goods the clearing price may in principle be the last accepted bid, the first rejected bid or any amount in between.

competition in supply functions [Klemperer and Meyer, 1989, Vives, 2011]. Despite their importance and apparent simplicity, we have a limited understanding of equilibrium behavior in these auctions when bidders have private information.

Analysis of these auctions is complicated by the multidimensional structure of bids, and the cross-dimensional information coupling that this implies. One such source of complication, identified by Vickrey [1961], is the fact that a bid on one unit may influence the price paid on prior units. A straightforward consequence is that bidding up to one's value on every marginal unit is weakly a dominated strategy — bidders engage in demand reduction [Ausubel et al., 2014].[2] The degree of demand reduction for a single bid may depend on the how many prior units a bidder has bid on as well as on the bidder's marginal values for those units. A competing bidder's inferences about the probability of winning against each bid must take these factors into account. To avoid these difficulties in models with privately informed bidders, authors have restricted attention to cases in which bidders demand exactly two goods [Ausubel et al., 2014, Engelbrecht-Wiggans and Kahn, 1998],[3] or to divisible-good models in which bidders have linear demands determined by normally distributed intercepts [Kyle, 1989, Vives, 2011].

We present a new approach towards modeling this auction in which bidders have multi-dimensional private information about their demand for any number of units of an indivisible good. The model allows one to flexibly specify each bidder's expected number of units demanded at each price, referred to as a bidder's mean demand curve, while imposing restrictions on the distribution of realized demand curves about this mean demand curve. We show that this model, applied to the last-accepted-bid uniform-pricing rule, is tractable and yields equilibrium behavior with several desirable properties, especially when compared to existing models.

We first show that with two bidders the equilibrium bids for *each* marginal unit take the form of bids from an asymmetric first-price auction for a *single* unit. If the two bidders' demands are symmetric, the bid curves can be solved for in closed form as in the first-price auction. Whereas the modern literature on uniform-price auctions — and multi-unit auctions more gen-

[2] Intuitively, a bidder reduces his bid below his demand curve for the same reason a monopolist's marginal revenue curve falls below its demand curve.

[3] With two goods and a pricing rule that specifies that the clearing price is equal to the first rejected bid, bidders have a weakly dominant strategy to bid their marginal value on the first unit. This leaves one bid per bidder to be determined in a natural class of equilibria.

erally — emphasizes differences between equilibrium bidding strategies in a multi-unit auction and a single-unit auction, we find a close connection between our model of the uniform-price auction and the first-price auction.[4] An immediate implication is that many of the results from the first-price auction literature translate directly to our environment. For example, we are able to draw a connection between the relationship between two bidders' mean demand curves and how aggressively they bid in the auction. By extending the work of Maskin and Riley [2000] we can classify the mean demand curves as "strong" or "weak" and translate these demand curves to bidding aggressiveness.

First-price bidding equilibria satisfy two important properties that are not typically satisfied in multi-unit auctions: they are unique, and they are separating in the sense that values can be inferred from bids. Both properties are valuable for empirical or computational work. It is immediate that the two-bidder equilibrium we identify is separating. Using techniques from the first-price auction literature we prove that it is unique in the space of separating strategies. With more than two bidders, equilibrium strategies in the last-accepted-bid uniform-price auction can no longer be directly identified with a corresponding first-price equilibrium, but we show that the properties of uniqueness and separation continue to hold with more than two bidders.

We compare the last-accepted-bid auction to the commonly-analyzed first-rejected-bid uniform-price and the pay-as-bid rules. We extend known inefficiency results in both the first rejected bid and pay-as-bid auctions (see, e.g., Ausubel et al. [2014]) to our model of bidder values. This provides a clear contrast with the unique equilibrium in the last accepted bid auction with symmetric distributions: this equilibrium is always efficient, while neither the first-rejected-bid nor the pay-as-bid auctions are efficient. We expose a connection between this inefficiency and the tractability problems which are known to complicate the first-rejected-bid and pay-as-bid auctions. In these two auctions, we show that information is pooled in all well-behaved equilibria. It is therefore impossible to achieve efficiency, and solving analytically for equilibrium involves the determination of pooled intervals, which imply regions over which the first-order conditions cannot be

[4]Historically, the literature on multi-unit auctions assumed (without analysis) they were strategically analogous to single-unit auctions; see, e.g., Friedman [1991]. Our results should not be mistaken for a return to this view. As stated above, our results suggest a strategic connection between single-unit first-price auctions and multi-unit last accepted bid auctions.

naïvely applied.[5] This provides a clear contrast with the tractable unique equilibrium we find in the last-accepted-bid auction. Last, we show that the low-revenue (collusive-seeming) equilibria which are known to plague the first-rejected-bid auction cannot be supported in the last-accepted-bid auction. These points together provide straightforward justification for employing the last-accepted-bid pricing rule.

Equilibrium strategies depend critically on the clearing price charged by the seller. Although it may be true that for a fixed set of strategies the bidders' payoffs are not affected much by choosing the last accepted bid or the first rejected bid as the clearing price, our results show that the choice of clearing price does have a significant effect on equilibrium strategies. The underlying reason is that the equilibrium bids are determined by conditioning on the low probability event that that particular bid is selected as the clearing price, and focusing on these events, our analysis makes clear that whether the clearing price is the last accepted bid or the first rejected bid has significant implications for how the bid is chosen.[6] Since our model accommodates any number of units, the equilibrium in our model provides a natural equilibrium selection in the divisible goods case, where the equilibrium need not be unique.

Pedagogically, these results suggest the care be paid to the selection of salient features of equilibrium in auction models. For example, bidders report truthfully in the canonical equilibrium of a second-price single-unit auction; it is known that the optimality of truthful reporting does not extend to multi-unit first-rejected bid pricing (see above, and also Back and Zender [1993], Engelbrecht-Wiggans and Kahn [1998], Wang and Zender [2002], and Ausubel et al. [2014] among many others).[7] It is also known that the intuitive behavior in a single-unit first-price auction does not translate to multi-unit discriminatory auctions, in spite of bidders paying their bids in

[5] In the first-rejected-bid auction, pooling arises for relatively low valuations for which the marginal gain associated with an increase in winning probability is outweighed by the increased probability of setting the market price. In the pay-as-bid auction, pooling arises due to the constraint that bids be weakly decreasing while agents would sometimes prefer to submit nonmonotone (or even weakly increasing) bid functions. The incentives underlying pooling behavior are distinct in these two auctions, but they imply similar issues for tractability.

[6] This distinction is relevant until the number of units becomes so large that the environment approaches a divisible good model in which the distinction between last accepted and first rejected is (typically) inconsequential.

[7] In particular, the optimality of truthful reporting in a second-price single-unit auction derives from its equivalence to a Vickrey auction. With multiple units this equivalence evaporates.

both settings (see, e.g., Woodward [2016]). Our results show strategic equivalence between single-unit first-price auctions and multi-unit last-accepted-bid auctions, suggesting that the strategically salient feature of these auctions is the selection of the highest market-clearing price. To our knowledge this has gone unaddressed in the literature.

The remainder of the paper is organized as follows. In Section 2, we analyze an example with two bidders and two goods to preview the main results. Section 3 introduces the general model. In Section 4, we characterize equilibrium in the last-accepted-bid uniform-price auction, and in Section 5 we present results on the separability and uniqueness of these equilibria. Section 6 provides contrasting results for the first-rejected-bid uniform-price auction and the pay-as-bid auction, while Section 7 concludes.

2 Leading Example: 2 Bidders and 2 Goods

We begin with a simple example of our model. There are two bidders, each with demand for (up to) two units. An auctioneer is selling two units in a multi-unit auction: he solicits weakly decreasing demands for each unit from each of the bidders, and awards the two units to the agent(s) submitting the two highest bids.

Bidders have independent private values: bidder i's value for her k^{th} unit is v_k^i. For each bidder, v^i is determined by ordering two independent draws from a $\mathcal{U}(0,1)$; in particular, v_k^i is (marginally) distributed according the k^{th} order statistic of a uniform distribution on $[0,1]$, $v_k^i \sim \mathcal{U}_{(k)}(0,1)$. Denote the inverse bid functions mapping bids to values by φ_k^i.[8]

Denote the marginal distribution of v_k^i by $F_{(k)}$. Because values are distributed as order statistics,

$$
\begin{aligned}
F_{(1)}(x) &= x^2, \\
F_{(2)}(x) &= 2x - x^2.
\end{aligned}
$$

We consider two payment rules. In *last accepted bid* (LAB), bidders pay the second-highest bid submitted for each unit they receive. In *first rejected bid* (FRB), bidders pay the third-highest bid submitted for each unit they

[8]In this analysis we elide some technical details and focus on well-behaved strategies; in particular, bids are strictly increasing in value and are thus invertible, and tiebreaking is a probability-zero event, so there is no concern about allocations when bidders submit the same bid.

receive.[9] We defer the discussion of pay-as-bid auctions, in which bidders pay their submitted bid for each unit they receive, until later in the paper.

Several statistical events are of importance in the pricing rules we investigate. Derivations of the associated probabilites may be found in the Appendix.

2.1 Last Accepted Bid

In LAB, three statistical events are salient. First, bidder i can win 2 units; this occurs when $b_2^i \geq b_1^{-i}$. Second, bidder i can win 1 unit while bidder $-i$ sets the price; this occurs when $b_1^i \geq b_1^{-i} > b_2^i$. Third, bidder i can win 1 unit and set the price; this occurs when $b_1^{-i} > b_1^i \geq b_2^{-i}$.[10]

With these events, interim utility in LAB can be expressed as

$$\begin{aligned} u^i\left(b^i; v^i\right) = &\left(v_1^i + v_2^i - 2b_2^i\right) \Pr\left(b_2^i \geq b_1^{-i}\right) \\ &+ \left(v_1^i - \mathbb{E}\left[b_1^{-i} \middle| b_1^i \geq b_1^{-i} > b_2^i\right]\right) \Pr\left(b_1^i \geq b_1^{-i} > b_2^i\right) \\ &+ \left(v_1^i - b_1^i\right) \Pr\left(b_1^{-i} > b_1^i \geq b_2^{-i}\right). \end{aligned}$$

As we show in Appendix A, in a symmetric equilibrium the agents' first-order conditions are given by

$$2\left(\varphi_1(b) - b\right)\left(1 - \varphi_2(b)\right) d\varphi_2(b) - \left(2\varphi_2(b) - \varphi_2(b)^2 - \varphi_1(b)^2\right) = 0; \quad \text{(unit 1)}$$
$$\left(\varphi_2(b) - b\right)\varphi_1(b)\, d\varphi_1(b) - \varphi_1(b)^2 = 0. \quad \text{(unit 2)}$$

These equations imply that in equilibrium,

$$\begin{aligned} \varphi_1(b) = \varphi_2(b) &= 2b, \\ b_1(v) = b_2(v) &= \frac{1}{2}v. \end{aligned}$$

That is, equilibrium in LAB is exactly equilibrium in a standard first-price auction for a single unit. It is immediate to verify that this equilibrium is efficient.

[9] As we will discuss later, LAB and FRB correspond to the highest and lowest (respectively) market-clearing prices in a Walrasian market with inelastic supply and demands given by the submitted bids.

[10] There is also a fourth relevant event, that bidder i wins zero units. Because this yields 0 utility, it is of no consequence to the formal analysis.

2.2 First Rejected Bid

In FRB, three statistical events are salient. First, bidder i can win 2 units; this occurs when $b_2^i \geq b_1^{-i}$. Second, bidder i can win 1 unit and set the price; this occurs when $b_1^{-i} > b_2^i \geq b_2^{-i}$. Third, bidder i can win 1 unit while bidder $-i$ sets the price; this occurs when $b_1^i \geq b_2^{-i} \geq b_2^i$.

With these events, interim utility in FRB can be expressed as

$$\begin{aligned} u^i\left(b^i; v^i\right) = &\left(v_1^i + v_2^i - 2\mathbb{E}\left[b_1^{-i}\middle| b_2^i \geq b_1^{-i}\right]\right) \Pr\left(b_2^i \geq b_1^{-i}\right) \\ &+ \left(v_1^i - b_2^i\right) \Pr\left(b_1^{-i} > b_2^i \geq b_2^{-i}\right) \\ &+ \left(v_1^i - \mathbb{E}\left[b_2^{-i}\middle| b_1^i \geq b_2^{-i} > b_2^i\right]\right) \Pr\left(b_1^i \geq b_2^{-i} > b_2^i\right). \end{aligned}$$

As we show in Appendix A, in a symmetric equilibrium the agents' first-order conditions are given by

$$\left(\varphi_1(b) - b\right)\left(2 - 2\varphi_2(b)\right) d\varphi_2(b) = 0; \quad \text{(unit 1)}$$

$$2\left(\varphi_2(b) - b\right)\varphi_1(b)\, d\varphi_1(b) - \left(2\varphi_2(b) - \varphi_2(b)^2 - \varphi_1(b)^2\right) = 0. \quad \text{(unit 2)}$$

The first-order condition with respect to the bid for the first unit, b_1^i, confirms the intuiton that truthful reporting is a weakly dominant strategy. This follows from standard second-price auction logic: the bid for the first unit is never the clearing price (when the agent wins) so it is effectively costless to increase the bid.[11]

In an equilibrium in which agents bid truthfully for their initial units the first-order condition with respect to the second-unit bid is no longer (meaningfully) a differential equation: $\varphi_1(b) = b$, and hence $d\varphi_1(b) = 1$. Subtituting through, symmetric equilibirum bids for the second unit must solve

$$2\left(v_2 - b\right) b - \left(2v_2 - v_2^2 - b^2\right) = 0.$$

This is a negative quadratic in b, with a solution of

$$b = v_2 \pm \sqrt{2v_2^2 - 2v_2}.$$

Since $v_2 \in [0,1]$, it must be that $2v_2^2 - 2v_2 \leq 0$ (and this inequality is strict when $v_2 \notin \{0,1\}$); then the negative quadratic has no real zeroes, and the first-order condition with respect to b_2 is negative everywhere. It follows that $b_2 = 0$ indentically, independent of v_2.

[11] Our technical results show that information confounding in equilibrium is independent of whether first-unit bids are truthful; it follows immediately that FRB equilibria are never efficient. We conjecture that revenue-dominance of LAB over FRB holds across all FRB equilibria, but we do not formally demonstrate this result.

2.3 Comparison

Because bids in the LAB auction are independent of the unit for which they are submitted—that is, because $b^i_k(v) = v_k/2$ for both units—outcomes are efficient. This contrasts strongly with the results of the FRB auction. Because second-unit bids are always zero, inefficient outcomes will arise whenever one agent's value for her first unit is below the other agent's value for her second unit (in this example, this probability is 1/3).

Due to the linearity of bids in the LAB auction, expected revenues may be simply computed as half of the expected second-highest draw from four draws of a uniform distribution. Aggregate expected revenue is then 3/5, and per-unit expected revenue is 3/10. This again contrasts strongly with the expected revenue of the FRB auction. Because second-unit bids are always zero, the clearing price is always zero. Then aggregate and per-unit expected revenues are zero.

In the remainder of this paper we demonstrate that certain of these properties generalize. When market demand satisfies a simple algebraic condition ("market balance") there is an equilibrium of the LAB auction in which bids are independent of the unit for which they are submitted, implying efficiency. We prove generally that there is a unique well-behaved[12] equilibrium in the LAB auction, establishing efficiency of a natural outcome of the auction. Contrariwise, we demonstrate that there is always information confounding and some degree of pooling in the FRB auction, thus all outcomes of the FRB auction are inefficient. We also demonstrate that the FRB auction always admits an equilibrium with expected revenue given by the reserve price (times the quantity sold), while the LAB auction admits no such equilibrium.

3 Model

An auctioneer sells m units of a homogeneous good to n risk-neutral bidders who, with probability 1, have strictly positive aggregate demand for at least m units. Bidder i values units according to the ordered realizations of m_i independent draws from the absolutely continuous distribution $F^i : [0,1] \to [0,1]$ with density f^i. The ordering ensures that marginal values are weakly declining for every realization. For example, the bidder's marginal value for the first unit is the first order statistic of m_i independent draws from F^i. When bidders are "symmetric", $F^i = F$ for all i and some F. We denote

[12]The proper notion of well-behavedness is defined later.

the ordered vector of bidder i's valuations by v^i, so that v^i_k is her value for her k^{th} unit. By definition, $v^i_1 \geq \cdots \geq v^i_{m_i}$. For simplicity we sometimes reference the "$n \times m$ case", in which n symmetric agents each have demand for m units, and m units are available in the auction. When $m = \sum_{j \neq i} m_j$ for any bidder i, we say that the market is *balanced.*

We consider sealed-bid auctions, where bidders submit weakly decreasing demand vectors to the auctioneer. Bidder i submits a weakly-positive demand vector b^i, so that b^i_k is her bid for her k^{th} unit. Where helpful, we will take a mechanism design approach and consider bids as functions of bidders' private values, $b^i_k \equiv b^i_k(v^i)$. Without a reserve price, the auctioneer allocates the available units to the m highest bids.[13,14,15] Denote the maximum and minimum market-clearing prices by $\overline{p}$ and $\underline{p}$, respectively, where

$$\begin{aligned}\overline{p} &= \min\left\{p : \#\left\{(i,k) : b^i_k \geq p\right\} \leq m\right\},\\ \underline{p} &= \max\left\{p : \#\left\{(i,k) : b^i_k \geq p\right\} \geq m\right\}.\end{aligned}$$

Each bidder is risk-neutral and her utility is quasilinear in payments. Conditional on allocation q_i and payment t_i, bidder i's ex post utility is

$$u^i\left(q_i, t_i; v^i\right) = \left[\sum_{k=1}^{q_i} v^i_k\right] - t_i$$

We focus most of our attention on the *last accepted bid* (LAB) uniform-pricing rule, in which each bidder pays the same price for each unit she obtains, and this price is equal to the m^{th} highest bid; this is equivalent to clearing the market at $\overline{p}$, the highest market-clearing price, $t_i(q_i) = \overline{p}q_i$. For mechanism comparisons, we also discuss the *first rejected bid* (FRB) uniform-pricing rule, where the per-unit price is the $(m+1)^{\text{th}}$ highest bid

[13] Bid monotonicity is a constraint typically observed in practice. However, under the assumption that the auctioneer accepts bids in decreasing order bid monotonicity is also a simplifying assumption that can be made without loss of generality.

[14] Where the m^{th} highest bid is not well-defined some form of tiebreaking or rationing is necessary. Because the tiebreaking rule is not of importance to our analysis, we leave it unspecified. This point has been noted in the multi-unit and divisible-good auction literature; see, e.g., Häfner [2015].

[15] When a nontrivial reserve price is present, bids are accepted in decreasing order until either all m units are allocated or there are no remaining bids weakly above the reserve price. For the most part our results are not meaningfully affected by the presence or absence of a reserve price (in light of what is known of behavior in single-unit auctions with reserve prices), so it is natural to ignore reserve price to avoid unnecessary technicalities.

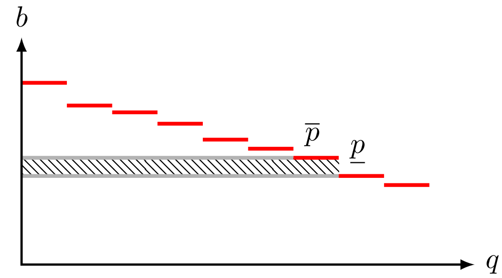

Figure 1: Maximum and minimum market-clearing prices displayed on an aggregate demand curve, $D(q) = \inf\{p : \#\{(i,k) : b^i_k \geq p\} < q$, when $m = 7$ units are available.

(equivalent to the lowest market-clearing price), $t_i(q_i) = \underline{p}q_i$, and the *pay-as-bid* (PAB) pricing rule, in which for each unit a bidder receives she pays her bid for this specific unit, $t_i(q_i) = \sum_{k=1}^{q_i} b^i_k$.

Market clearing implies that bidder i receives unit k if and only if her opponents receive (in aggregate) less than $m-k+1$ units.[16] It is helpful to consider bidder i competing for her k^{th} unit against the aggregate demand of her opponents for $m-k+1$ units. Let H^i_{m-k+1} be the marginal distribution of her opponents' $m-k+1^{\text{th}}$ highest bid, and let h^i_{m-k+1} be the associated density (where well-defined).

3.1 Matching Demand Curves

There is a natural interpretation of our "order statistics" model in terms of bidders' mean demand curves in the following sense. Fixing a uniform price p, the expected number of units demanded by bidder i is $(1 - F^i(p))m_i$.[17] The specification of the mean demand curve is therefore flexible because F^i is arbitrary, while the distribution of demand curves about the mean is determined by the properties of the order statistic model.[18]

[16] With a reserve price the "only if" is still valid, but the "if" may fail. Nonetheless the competition faced for unit k is against opponents' aggregate demand for $m-k+1$ units.

[17] For a fixed price, the number of units demanded out of a maximum of m is a random variable with a binomial distribution with probability of "success" given by $1 - F^i(p)$.

[18] For example, the observations in Footnote 17 imply that the variance of the number of units demanded at price p must be $F^i(p)(1-F^i(p))m_i$, or large for intermediate prices and small for prices near 0 or 1.

4 Equilibrium of Last Accepted Bid

We first derive equilibrium bidding strategies in the last-accepted-bid auction with symmetric demands when the balanced market condition holds, and show that they have closed form representations.

4.1 Symmetric Bidders in a Balanced Market

Recall that a balanced market is one where $m = \sum_{j \neq i} m_j$. This implies that each bidder faces exactly m bids from opponents in equilibrium. The case where two bidders each demand m units is one such example.

Symmetric means that each bidder's k^{th} unit is distributed according to the k^{th} order statistic from the same distribution F (i.e., $v_k^i \sim F_{(k)}$ for each i). We use $Y_{(k)}$ to denote a random variable that is the k^{th} order statistic from m independent draws from F. Building on the example in Section 2, suppose that each opposing bidder shades her bid consistently for every unit in the sense that there is some increasing $b(\cdot)$ such that for all agents j and all units k, $b_k^j = b(v_k^j)$. Let $\varphi(b)$ be the inverse of $b(v)$.[19] The distribution of the opponent's k^{th} bid is therefore $F_{(k)}(\varphi(b))$.

If bidder i places the bid $b_k^i = b(v_k^i)$ on her k^{th} unit, under the last accepted bid rule, bidder i wins exactly k units and pays $b(Y_{(m-k)})$ if and only if $v_k^i \geq Y_{(m-k)} \geq v_{k+1}^i$. On the other hand, bidder i wins k units and pays $b(v_k^i)$ if and only if $Y_{(m-k)} \geq v_k^i \geq Y_{(m-k+1)}$. If $k > 1$, bidder i's bid on the k^{th} unit also affects the probability of winning exactly $k-1$ units when the realization of values is such that $v_{k-1}^i \geq Y_{(m-k-1)} \geq v_k^i$. These three events account for all the ways in which the bid on the k^{th} unit affects bidder i's payoff. To derive a necessary condition for b to be an equilibrium consider a bid $b(v') \in (b(v_{k-1}^i), b(v_{k+1}^i))$ and the associated contribution to bidder i's payoff.

$$
\begin{aligned}
u_k^i\left(b; v^i\right) = & \int_{v_{k+1}^i}^{v'} \left(\sum_{\ell=1}^{k} v_\ell^i - kb\left(x\right) \right) f^{(m-k)}\left(x\right)\, dx \\
& + \left(\sum_{\ell=1}^{k} v_\ell^i - kb\left(v'\right) \right) \left(F_{(m-k+1)}\left(v'\right) - F_{(m-k)}\left(v'\right)\right) \\
& + \int_{v'}^{v_{k-1}^i} \left(\sum_{\ell=1}^{k-1} v_\ell^i - (k-1)\, b\left(x\right) \right) f_{(m-k-1)}\left(x\right)\, dx
\end{aligned}
$$

[19] We show later that this inverse exists.

The derivative of this expression with respect to v' evaluated at v_k^i is

$$\begin{aligned}
&\left(v_k^i - b\left(v_k^i\right)\right) f_{(m-k+1)}\left(v_k^i\right) - kb'\left(v_k^i\right)\left(F^{(m-k+1)}\left(v_k^i\right) - F^{(m-k)}\left(v_k^i\right)\right) \\
&= \binom{m}{m-k} k\left(1 - F\left(v_k^i\right)\right)^{m-k} F\left(v_k^i\right)^{k-1}\left[f\left(v_k^i\right)\left(v_k^i - b\left(v_k^i\right)\right) - b'\left(v_k^i\right) F\left(v_k^i\right)\right].
\end{aligned} \tag{1}$$

One can check that the familiar solution to the symmetric first-price auction for a single unit, $b(v) = \frac{1}{F(v)} \int_0^v x f(x)\, dx$ makes this expression zero, and this is in fact the equilibrium mapping from marginal values to bids.

Proposition 1. *If marginal values for each bidder are the order statistics from independent draws from F the equilibrium bids for bidder i in the last-accepted-bid auction are*

$$\mathbf{b}^i(\mathbf{v}_i) = \left(\frac{1}{F(v_k^i)} \int_0^{v_k^i} x f(x)\, dx\right)_{k \in \{1,..,m_i\}}.$$

Proof. From the discussion preceding the proof, it is clear that the candidate equilibrium bid, $b(v_k^i)$, satisfies the k^{th} first-order condition. Standard arguments establish that the partial derivative is negative (positive) for $b(v')$ when $v' > v_k^i$ ($v' < v_k^i$), which also means that the objective must be lower at the end points, $b(v_{k-1}^i)$ and $b(v_{k+1}^i)$, given by the monotonicity constraint. Furthermore observe that all cross-partial derivatives are zero, from which it follows that the second-order conditions are satisfied.

For $k = 1, \ldots, m$, the relevant components of the objective are respectively

$$\int_{v_2^i}^{v'} \left(v_1^i - b\left(x\right)\right) f_{(m-1)}\left(x\right)\, dx + \left(v_1^i - b\left(v'\right)\right)\left(F_{(m)}\left(v'\right) - F_{(m-1)}\left(v'\right)\right)$$

and

$$\left(\sum_{j=1}^{m} v_j^i - mb\left(v'\right)\right) F_{(1)}\left(v'\right) + \int_{v'}^{v_{m-1}^i} \left(\sum_{j=1}^{m-1} v_j^i - (m-1)\, b\left(x\right)\right) f_{(1)}\left(x\right)\, dx.$$

Similar arguments establish the optimality of setting $b_1^i = b(v_1^i)$ and $b_m^i = b(v_m^i)$. $\square$

This equilibrium is efficient, and hence standard arguments imply that the expected payment should be equal to the Vickrey payment. To see this, consider the event that $b(v_k^i)$ is the last accepted bid (i.e., $Y_{(m-k)} \geq v_k^i \geq Y_{(m-k+1)}$). In this event the bidder pays $kb(v_k^i)$, which is the expected payment made for k units in a Vickrey auction conditional on this event because

$$\begin{aligned}
&\sum_{j=1}^{k} \mathbb{E}\left[Y_{(m-k+j)}\middle| Y_{(m-k)} \geq v_k^i \geq Y_{(m-k+1)}\right] \\
&= \sum_{j=1}^{k} \mathbb{E}\left[Y_{(j:k)}\middle| v_k^i \geq Y_{(1:k)}\right] \\
&= k\mathbb{E}\left[Y | v_k^i \geq Y\right] = kb(v_k^i),
\end{aligned} \tag{2}$$

where the notation $Y_{(j:k)}$ denotes the j^{th} highest value out of k independent draws from F. To understand the first equality, observe that conditional on the event $Y_{(m-k)} > v_k^i > Y_{(m-k+1)}$ the first $m-k$ random variables provide no additional information about the last k random variables, so the expectation reduces to one involving just the last k. Recall that in the Vickrey auction a bidder who wins k units in this environment would be required to pay the sum of the k rejected bids made by the opponents. In other words, the bids in this equilibrium are set so that the expected payment equals the expected Vickrey auction payment conditional on the event that the bid determines the payment.

4.2 The Asymmetric $2 \times m$ Case

When marginal values for two bidders, who each demand all m units, are drawn from different distributions, the equilibrium will in general no longer be efficient or have closed-form expressions, as is the case in asymmetric first-price auctions for a single good. However, given that first-order conditions in this auction take forms very similar to that of the first-price auction for a single good, many of our earlier results carry over.

Most of the literature on asymmetric first-price auctions focuses on the two-bidder case. An analogous case in this model is set up as follows. Suppose that there are two bidders, $i = 1, 2$, who each value marginal units according to m draws each from F^i, where $F^1 \neq F^2$. Suppose that i's bids for each marginal unit are determined by the increasing, differentiable function $b^i(v_k^i)$ with inverse $\varphi^i(b)$. Then the event that i's bid on the k^{th} unit, b_k^i, is selected as the last accepted bid occurs if and only if $Y_{(m-k)} \geq \varphi^{-i}(b_k^i) \geq Y_{(m-k+1)}$. Accounting for the other two events influenced by the choice of b_k^i (see Section 4.1), the first-order condition associated with b_k^i if $b_k^i \in (b_{k+1}^i, b_{k-1}^i)$ is

$$\binom{m}{m-k} k (1 - F^{-i}(\varphi^{-i}(b_k^i)))^{m-k} F^{-i}(\varphi^{-i}(b_k^i))^{k-1} \times \\ \left\{ \varphi^{j\prime}(b_k^i) f^{-i}(\varphi^{-i}(b_k^i))(v_k^i - b_k^i) - F^{-i}(\varphi^{-i}(b_k^i)) \right\} = 0 \quad (3)$$

which reduces to the expression studied in Maskin and Riley [2000]. Suppose $b^1(v)$ and $b^2(v)$ are the equilibrium bid functions from the first-price auction involving two bidders with corresponding value distributions F^1 and F^2, then it is immediate that setting $b_k^i = b^i(v_k^i)$ will satisfy bidder i's first-order condition for good k when $\mathbf{b}^i(\mathbf{v}^i) = \left(b^i(v_k^i)\right)_{k \in \{1,\ldots,m\}}$.

Proposition 2. *With two bidders whose m marginal values are the order statistics from F^1 and F^2, if $b^1(v)$ and $b^2(v)$ are the equilibrium bid functions in the first-price auction for a single unit with two bidders whose values are distributed according to F^1 and F^2, then the strategies $\mathbf{b}^i(\mathbf{v}^i) = \left(b^i(v_k^i)\right)_{k \in \{1,..,m\}}$ constitute an equilibrium of the last-accepted-bid auction.*

Proof. Analogous to the proof of Proposition 1. □

Given the relation between equilibrium bidding in the last-accepted-bid model and bidding in asymmetric first-price auctions, a number of results follow immediately. Instead of exhaustively listing them here, we emphasize their interpretation in this model. Recall that we may interpret the function $(1 - F^i(p))m$ as bidder i's mean demand curve. It follows from the previous section, that $(1 - F^i(\varphi^i(b)))m$ represents the mean number of bids placed by bidder i that exceed b in equilibrium, referred to as the mean quantity demanded in equilibrium. Bidder $-i$'s mean residual supply curve is therefore $F^i(\varphi^i(b))m$, which is proportional to the equilibrium bid distribution of a bidder with type distribution F^i in a first-price auction.

The stochastic dominance properties used in the asymmetric first-price auction literature have immediate analogues to properties of the mean demand curves in this model. For example, bidder i having weakly higher mean demand than bidder $-i$ at each price is equivalent to F^i first-order stochastically dominating F^{-i}. An implication from the first-price auction literature is that bidder i's mean quantity demanded weakly exceeds bidder $-i$'s in equilibrium [Kirkegaard, 2009, Corollary 1]. The stronger distributional ordering property of reverse hazard rate dominance can be stated as follows.

Definition 1 (Reverse Hazard Rate Dominance)**.**

$$F \succeq_{rh} G \iff \frac{d}{dx} \frac{F(x)}{G(x)} \geq 0, \, \forall x.$$

When F and G admit densities at x, this implies $f(x)/F(x) \geq g(x)/G(x)$. If $F^i(x)m$ is the mean residual supply curve that bidder i would present to bidder $-i$ if she were to bid her value for each unit, then $xf^i(x)/F^i(x)$ is the elasticity of that supply curve. The reverse hazard rate condition can then be interpreted as requiring that these elasticities are ordered. From Proposition 3.5 of Maskin and Riley [2000] we can therefore conclude that this ordering of elasticities is sufficient to order the bid curves of the bidders, meaning $F^i \succeq F^{-i}$ implies $b^i(v) < b^{-i}(v)$ or $\mathbf{b}^i(\mathbf{v}) < \mathbf{b}^{-i}(\mathbf{v})$. This is the well-known "weakness leads to aggression" result.

Finally, we make one more connection to work on investment incentives in single unit auctions. In their Proposition 3 Arozamena and Cantillon [2004] show that if one bidder is given the opportunity to "upgrade" their type distribution ex ante by making it stronger with respect to hazard-rate dominance, the investment incentives are stronger in the second-price auction than in the first-price auction. Furthermore, their Proposition 4 shows that investment incentives are optimal in the second-price auction. Upgrading the distribution has a natural interpretation in our model. It is equivalent to a bidder in our model investing to increase her mean demand curve in such a way as to weakly increase the elasticity of the mean residual supply curve at every point. From the Arozamena and Cantillon [2004] results we get immediate comparisons of the investment incentives in the last-accepted-bid uniform-price auction to those in the Vickrey auction, which is the extension of the second-price auction to this environment.

4.3 The General Case

If the market is either balanced and bidders have symmetric demands or there are two bidders with asymmetric demand for all units, we can identify equilibrium strategies with a corresponding first-price auction. This is no longer true in the general case where the market is either unbalanced or there are more than two asymmetric bidders. A common property of the equilibria in both of the previous sections is that there exists a univariate function which bidder i uses to determine the bids on all of his marginal units from their marginal values. In general this property, which allows for the reduction to a first-price auction, does not hold in equilibrium, and bidders may shade their bids on marginal units differently depending on the unit to which the bid is on.

Despite not being able to pin down equilibrium strategies in the general case, we show in this section that we can utilize techniques from the first-price auctions literature to establish that some key properties still hold. For example, we present an unbalanced, symmetric demand case in which we can prove a uniqueness result for equilibrium strategies using an argument that closely resembles the uniqueness argument typically given for the equilibrium of a first-price auction.

First, we provide an existence result for the general model.

Proposition 3. *With $n \geq 2$ bidders $i \in \{1, \ldots, n\}$, where bidder i's m_i marginal values are the order statistics from the distribution F^i, the last-accepted bid uniform price auction admits a pure-strategy Bayesian Nash equilibrium.*

Proof. This follows from Corollary 5.2 in Reny [2011].[20] □

5 Properties of the LAB Equilibrium

The previous section shows that there is a close connection between the equilibrium of the first-price auction and that of the LAB auction. In this section we discuss the properties of these equilibria with a particular focus on whether the equilibrium is separating and whether it is unique.

We divide the analysis into two cases. In the first, we discuss the case with two bidders and m units where the bidder i's demand curve is generated from m ordered draws from some F^i. The second considers the case where we have n symmetric bidders bidding for two units and each bidder has a demand curve generated from two ordered draws from a common F.

5.1 Separation in the $2 \times m$ Case

A formal definition of what we mean by separating bids is the following.

Definition 2 (Strictly separating bids)**.** *A bid function b^i is* strictly separating *if the inverse bid correspondence is at most single-valued; that is, for all type profiles v^i,*

$$\#\left\{v :\ b^i\left(v\right) = b^i\left(v^i\right)\right\} = 1.$$

In equilibria that are strictly separating bid curves preserve all information about the marginal values. In other words, bid curves are invertible given bid data — a useful property for empirical work. Because the monotonicity constraint on bid curves never binds, the optimization problem can

[20]Reny [2011] investigates the FRB auction. With regard to existence (although not, as argued above, the structure of equilibrium) the arguments do not change in a substantive way.

be solved bid-by-bid. This reduces the computational complexity of the bidder's problem as well as the complexity of computationally solving for equilibrium.

Observe that in the case with two bidders and m units the equilibria described in Section 4 satisfy this property, since the event that a bidder submits two marginal bids that are equal to one another has zero measure.

Corollary 1. *Suppose two bidders each with demand for m goods compete for m goods in a LAB uniform-price auction. There exists an equilibrium in which bids are strictly separating.*

In Section 6 we show that this property is not shared by either the FRB uniform-price auction or the PAB auction.

5.2 Uniqueness in the $2 \times m$ Case

Several authors have investigated the uniqueness of equilibrium bidding strategies in the first-price auction, notably Maskin and Riley [2003], Bajari [2001], and Lebrun [2006]. Their arguments for uniqueness are based on analyses of the system of differential equations in the inverse bid functions derived from first-order conditions. In our derivation of equilibrium for the LAB auction, we show that under the assumption that the opponent uses the same univariate bid function for each marginal unit we recover the same system of differential equations (e.g., see (3)). It follows that if the bidders are restricted to using the same bid function for each marginal unit the existing uniqueness results for the first-price auction apply in our setting.

In this section, we extend this result to show uniqueness over a larger set of strategies. We show that the equilibrium identified in the previous section is unique among all separating strategies. More precisely, we show that whenever the corresponding first-price auction admits a unique equilibrium, the equilibrium we have identified is unique among separating strategies.

The arguments for uniqueness in the first-price auction given in the literature typically use the same intermediate arguments.[21] First, one shows that the largest equilibrium bid (or smallest in the case of procurement) is the same for every bidder. Second, one defines a system of ordinary differential equations involving inverse bid functions. The equations in the system

[21] We refer to Lebrun [2006] for a discussion of uniqueness results in the first-price auction literature and the assumptions required to prove uniqueness. There is a unique equilibrium in the asymmetric first-price auction under fairly general conditions, but as argued in Lebrun [2006] some prior proofs have relied on unjustified uses of L'Hôpital's rule.

are shown to be necessary and sufficient for optimality and also to satisfy the Lipschitz condition at every bid excluding the lowest bid. The initial value problem starting from a particular highest bid therefore has a unique solution due to the fundamental theorem of ordinary differential equations. Third, one shows that if $\bar{b}$ and $\widetilde{b}$ are two initial values with $\bar{b} < \widetilde{b}$ then the solutions to the initial value problem using $\bar{b}$ are greater than those to the problem using $\widetilde{b}$ at every interior b. Finally, one shows with an additional assumption about the problem at the lowest bid that the second and third results imply that there can only be one highest bid yielding a solution that is also an equilibrium.

To establish uniqueness of the LAB equilibrium among separating strategies in our model, we follow the first two steps in the prior paragraph but then appeal to the uniqueness of the corresponding first-price auction solution to complete the proof. We restrict attention to separating strategies, because our argument relies on the analysis of a system of differential equations that is only valid for separating strategies. Allowing the monotonicity constraint to bind for arbitrary bids leads to a system of equations that is substantially more difficult to analyze.

The most important step in our argument is to establish that the highest bid submitted for any unit by any bidder in equilibrium is the same. This does not follow directly from the analogous argument in the first-price auction, although there are similarities. The added difficulty here arises from the facts that there is a monotonicity constraint on bids and that the probability that a bid on unit k wins depends on the distribution of two of the opponent's bids. We show that the highest equilibrium bid is the same for all units in Lemma 2, after proving an intermediate lemma next.

Lemma 1. *Suppose that a type-v^i bidder i submits a constant bid $b^i_{\{k,\ldots,k+a\}}$ for units $k,\ldots,k+a$ and let $b^i_l(v^i_l)$ and $b^i_s(v^i_s)$ with $l,s \in \{k,\ldots,k+a\}$ be respectively any of the bidder's largest and smallest unconstrained bids for these units. Then $b^i_l(v^i_l) > b^i_s(v^i_s)$ implies $b^i_l(v^i_l) > b^i_{\{k,\ldots,k+a\}} > b^i_s(v^i_s)$.*

Proof. The first-order condition for the constrained bid is

$$\sum_{j=k}^{k+a} \frac{\partial}{\partial b^i_j} U^i(b^i_{\{k,\ldots,k+a\}}; v^i) = 0, \tag{4}$$

or the sum of the unconstrained bid first-order conditions. Note that the objective is quasi-concave in each b^i_j. At the largest unconstrained bid, b^i_l, the first-order conditions for the other bids cannot be positive, due to quasiconcavity, and given $b_l^i(v_l^i) > b_s^i(v_s^i)$ at least one is negative. Therefore, at b_l^i, (4) is negative. A similar argument implies that (4) is positive at b_s^i. $\square$

Lemma 2. *In equilibrium, there is a $\bar{b}$ such that for all i and k, $\bar{b}_k^i = \bar{b}$.*

Proof. First, it cannot be that $\bar{b}_k^i = \bar{b}^i$ for all i and k but $\bar{b}^i \neq \bar{b}^{-i}$ because the type of bidder who submits the higher maximum bid could lower all of his bids and reduce his payment without reducing the probability of winning any units. Therefore if the lemma is false $\bar{b}_k^i > \bar{b}_{k+1}^i$ for some k and i. Let $\hat{k}$ to be the lowest k for which $\bar{b}_k^i > \bar{b}_{k+1}^i$.

We claim that $\bar{b}_{m-\hat{k}+1}^{-i} = \bar{b}_{m-\hat{k}}^{-i} = \bar{b}_{\hat{k}}^i$. It must be that $\bar{b}_{m-\ell+1}^{-i} \leq \bar{b}_\ell^i (= \bar{b}_{\hat{k}}^i)$ for all $\ell \leq \hat{k}$ because otherwise the type of bidder $-i$ placing these bids could weakly reduce all of these bids without reducing the probability of winning any of the items. If $\bar{b}_{m-\ell+1}^{-i} \leq \bar{b}_{m-\hat{k}+1}^{-i} < \bar{b}_{\hat{k}}^i$ for all $\ell \leq \hat{k}$, then bidder i should respond by reducing his maximum bids on the first $\hat{k}$ units for the same reason. Hence, $\bar{b}_{m-\hat{k}+1}^{-i} = \bar{b}_{\hat{k}}^i$. Now $\bar{b}_{m-\hat{k}+1}^{-i} = \bar{b}_{m-\hat{k}}^{-i}$ follows because otherwise $\bar{b}_{m-\hat{k}}^{-i} > \max\{\bar{b}_{m-\hat{k}+1}^{-i}, \bar{b}_{\hat{k}+1}^i\}$ and $\bar{b}_{m-\hat{k}}^{-i}$ can be reduced without lowering the probability of winning or violating monotonicity.

Finally, for $v_{m-\hat{k}}$ close to 1, $b_{m-\hat{k}}^{-i}(v_{m-\hat{k}}) \leq \bar{b}_{\hat{k}+1}^i < \bar{b}_{\hat{k}}^i = \bar{b}_{m-\hat{k}+1}^{-i}$. Lemma 1 then implies that the optimal choice of constrained bid for bidder 2 is strictly below $\bar{b}_{\hat{k}}^i$, which is a contradiction. $\square$

Having established that there is a common maximum bid for all units and bidders, we next describe the system of differential equations we evaluate. As with first-price auctions, the arguments are made simpler by writing the differential equations in terms of an unknown derivative with respect to a bid distribution (cf. Lebrun [2006]). Recall that in this section we are assuming that the bidders use separating strategies. This implies that $\varphi_k^i(b) \leq \varphi_{k+1}^i(b)$ for all k and i. Consequently, the distribution of the k^{th} bid of bidder i is $F_k^i(\varphi_k^i(b))$. Furthermore, bidder $-i$'s first-order condition with respect to his $(m-k+1)^{th}$ bid becomes

$$[\varphi_k^i]'(b) f_k(\varphi_k^i(b))(v_{m-k+1}^{-i} - b) - (m-k+1)\left(F_k^i(\varphi_k^i(b)) - F_{k-1}^i(\varphi_{k-1}^i(b))\right) = 0.$$

We create a system of $2m$ differential equations out of the first-order conditions for each bid by each bidder. Instead of writing the system in terms of unknown inverse bid functions, we write it in terms of unknown bid distributions as follows.

Definition 3. *Let* $H_k^i(b) \equiv F_k^i(\varphi_k^i(b))$, $H_0^i(b) \equiv 0$, $\varphi_{m-k+1}^{-i} \equiv [F_{m-k+1}^{-i}]^{-1} H_{m-k+1}^{-i}$, *and* $\bar{b} \in (0,1)$ *be given. Find* $(H_k^1, H_k^2)_{k=1,\ldots,m}$ *such that for all* $1 \leq k \leq m$, $i \in \{1,2\}$, *and* $b \in (0, \bar{b}]$

$$\begin{aligned} \frac{d}{db} H_k^i(b) &= \frac{m-k+1}{\varphi_{m-k+1}^{-i} - b} \left(H_k^i(b) - H_{k-1}^i(b) \right) \qquad (5) \\ H_k^i(\bar{b}) &= 1 \end{aligned}$$

This initial value problem involves a system of $2m$ equations in $2m$ unknown functions, $H_k^i(b)$. The next lemma establishes that an equilibrium of the LAB auction is necessarily a solution to this initial value problem.

Lemma 3. *Any equilibrium bid profile in separating strategies must satisfy* (5).

Proof. This is implied by continuity and differentiability of the bid distribution functions. These results are similar to the arguments familiar from the first-price auction, but we reproduce them here due to the changes in the agents' utility functions induced by shifting to a multi-unit model. We say that unit k is *opposed to* unit $m-k+1$, in the sense that agent i wins unit k if and only if agent $j \neq i$ wins unit $m-k+1$. Recall the separable utility representation for the LAB auction,

$$\begin{aligned} u^i(b; v) = \sum_{k=1}^{m} v_k H_{m-k+1}^{-i}(b_k) - \left(H_{m-k+1}^{-i}(b_k) - H_{m-k}^{-i}(b_k) \right) k b_k \\ - k \int_0^{b_k} x dH_{m-k}^{-i}(x) + (k-1) \int_0^{b_k} x dH_{m-k+1}^{-i}(x). \end{aligned}$$ [22]

First, there are no gaps in equilibrium bid distribution functions. If there is a gap in H_{m-k+1}^{-i}, then a bid for agent i's unit k strictly inside this gap induces no additional winning probability but incurs additional expected costs (vis a vis bidding the lower bound). It follows that any gaps in H_{m-k+1}^{-i} are shared by the opposing distribution H_k^i. Since there is no probability gain within the gap, for a bid to be placed at the upper end

[22] This expression appears to presuppose the differentiability of $dH_{k'}^{-i}$ for all k', however it is a re-expression of one in terms of well-defined conditional expectations; since we establish that in equilibrium the bid distributions are continuously differentiable this expression is ultimately correct. We do not presuppose the correctness of this expression, and avoid this potential circularity in our formal arguments.

of the gap there must be a mass point;[23] there are therefore identical mass points for the opposing units k and $m-k+1$. Identical mass points cannot arise for standard tiebreaking reasons, therefore this is not supportable in equilibrium.

Second, above the reserve price there are no mass points in equilibrium bid distributions (i.e., equilibrium bid distributions are continuous). Suppose that there is a mass point in H^{-i}_{m-k+1} at bid b, but no mass point in H^{-i}_{m-k}. Since bids are in general strictly below values[24] and there are no gaps in the bid distributions, there is a value v such that $b^i_k(v) = b - \varepsilon$ for any $\varepsilon > 0$. For ε small enough, a slight increase to $\tilde{b}^i_k(v) = b + \varepsilon$ yields a discrete jump in expected utility; this implies that gaps exist in response to mass points, and we have already established that gaps cannot exist. Otherwise, suppose that there are mass points in both H^{-i}_{m-k+1} and H^{-i}_{m-k} at b, so that the above logic does not apply. However, if this is the case, then there is a mass point in H^{-i}_{m-k}, the unit opposed to bidder i's unit $k+1$. Then the previous argument holds unless there is also a mass point in H^{-i}_{m-k-1}, and so on. Since there are no mass points in the degenerate distribution H^{-i}_0—the $H^{-i}_{m-\tilde{k}}$ corresponding to $\tilde{k} = m$—the original argument must hold for some unit, violating the no-gaps property established above.

Lastly, equilibrium bid distributions are differentiable above the reserve price. Suppose that H^{-i}_{m-k+1} is not differentiable at b while H^{-i}_{m-k} is. As is familiar, this implies the existence of a gap (in the case of an upward kink) or a mass point (in the case of a downward kink) in the opposing bid distribution for agent i's unit k. Since both of these have been ruled out above, this nondifferentiability is not possible. The case in which both H^{-i}_{m-k+1} and H^{-i}_{m-k} are nondifferentiable at b can be handled similar to the analysis of mass points above. Then equilibrium bid distributions must be differentiable.

Since equilibrium bid distributions are continuous and differentiable (above the reserve price), the first-order conditions must be satisfied in any equilibrium in separating strategies. □

Compare the problem in Definition 3 to the following corresponding one for the first-price auction.

[23] This analysis ignores the possibility that the support of the bid distribution above the gap is left-open. For a bid sufficiently close to this upper endpoint, the arguments are the same.

[24] This somewhat obvious point is proved explicitly in a Lemma in an earlier version of this paper.

Definition 4. *Let $H^i(b) \equiv F^i(\varphi^i(b))$ and $\bar{b} \in (0,1)$ be given. Find (H^1, H^2) such that for all $b \in (0, \bar{b}]$ and $i \in \{1,2\}$,*

$$\frac{d}{db}H^i(b) = \frac{H^i(b)}{\varphi^{-i} - b} \tag{6}$$
$$H^i(\bar{b}) = 1.$$

For an arbitrary $\bar{b}$, because (6) satisfies the Lipschitz condition for all $b \in (0, \bar{b}]$, the Fundamental Theorem of Ordinary Differential Equations (FTODE) implies there is a unique solution to the initial value problem in Definition 4. Furthermore, when there is a unique equilibrium in the first-price auction, a single such $\bar{b}$ yields a solution that also satisfies the boundary condition $H^i(\underline{b}) = F^i(\varphi^i(\underline{b})) = 0$, where $\underline{b}$ is the lowest equilibrium bid.

Since the system in (5) also satisfies the Lipschitz condition for all $b \in (0, \bar{b}]$, the FTODE implies that there is a unique solution to the problem in Definition 3 given a $\bar{b}$. But these two solutions must coincide in the sense that if (φ^1, φ^2) is a solution to the first-price auction problem by setting $\varphi^i_k = \varphi^i$ for all k and i and examining Equation (3) we find the unique solution to the initial value problem corresponding to the LAB auction as well. The final step is to observe that while Proposition 2 gives us that the equilibrium value of $\bar{b}$ generates equilibrium solutions to both problems a different $\bar{b}$ would generate a solution that is not an equilibrium of the first-price auction (by uniqueness) and cannot be an equilibrium of the LAB auction.

Proposition 4. *Consider the last-accepted-bid auction between two bidders whose m marginal values are the order statistics from F^1 and F^2 and the corresponding first-price auction involving two bidders with value distributions F^1 and F^2. If the equilibrium in the first-price auction is unique, then there is one equilibrium of the last-accepted-bid auction in which the bidders use separating strategies (i.e., ones in which the monotonicity constraint binds with probability zero).*

5.3 Separation in the $n \times 2$ Case

We turn next to the case with n symmetric bidders for two units. We do not have an explicit equilibrium characterization here, yet, as we show next, all symmetric equilibria must be separating.

Let $G_1(b)$ and $G_2(b)$ be the equilibrium distributions of *each bidder's* first and second bid. Define F_1^{-1} as the inverse of the distribution of the first order statistic from F. Define F_2^{-1} similarly for the second order statistic. After isolating $g_1 \equiv G_1'$ and $g_2 \equiv G_2'$, the first-order conditions require that

$$\begin{aligned} g_1(b) &= \frac{2}{n}\frac{G_1(b)}{\varphi_2(b)-b} \\ g_2(b) &= \frac{G_2(b)-G_1(b)}{\varphi_1(b)-b} - \frac{2(n-1)}{n}\frac{G_2(b)-G_1(b)}{\varphi_2(b)-b}, \end{aligned}$$

where we have used the shorthand $\varphi_1(b) = F_1^{-1}(G_1(b))$ and $\varphi_2(b) = F_2^{-1}(G_2(b))$. Note that $F_1^{-1}(G_1(b))$ and $F_2^{-1}(G_2(b))$ are simply the values for the first and second unit associated with a bid of b. Obviously, we must have $g_2(b) \geq 0$ in equilibrium. This requires that

$$\frac{1}{\varphi_1(b)-b} \geq \frac{2(n-1)}{n}\frac{1}{\varphi_2(b)-b}$$

which implies $\varphi_2(b) \geq \varphi_1(b)$ as long as $n \geq 2$ and that this inequality is strict when $n \geq 3$. Note that the monotonicity constraint implies that $G_2(b) \geq G_1(b)$ for all b.

Proposition 5. *With $n \geq 2$ symmetric bidders for two units, all symmetric equilibria have separating bids.*

Proof. See Appendix C. □

The following corollary is useful in the next section.

Corollary 2. *When $n \geq 3$, and equilibrium bids are monotone in values $b_1(1) > b_2(1)$.*

Corollary 2 exposes a distinction between the 2-bidder and n-bidder cases of the LAB auction. With only two bidders, a bid for unit k competes only against an opponent's bid for unit $m-k$; then distributions of bids for these units should have the same upper bound. With more agents this intuition fails. When there are only two units, for example, a bid for the second unit competes only against opponents' first-unit bids, while a bid for the first unit competes against opponents' bids for both the first and second units. Then it is no longer true that the bid distributions must be equal, only that the support of second-unit bids is a subset of the support of first-unit bids.

5.4 Uniqueness in the Symmetric $n \times 2$ Case

In the prior section we show that equilibria in the $n \times 2$ case must involve separating strategies. We evaluate the uniqueness of a symmetric equilibrium in the $n \times 2$ case in this section. Since the 2×2 case is covered by the previous analysis we assume that $n \geq 3$. Let $b_1(v_1)$ and $b_2(v_2)$ represent a candidate equilibrium, where $b_1(v) \geq b_2(v)$ for all $v \in (0,1)$. Denote the inverse bid functions by φ_1 and φ_2 respectively.

The argument closely resembles the uniqueness argument for the $2 \times m$ case given in Section 5.2, which in turn resembles uniqueness arguments given for asymmetric first-price auctions. A key difference that arises is that with more than two bidders it is no longer true that there is a common high bid for each marginal unit. One important implication of proving that there *is* a common high bid for each marginal unit is that the initial conditions for the system of differential equations derived from the first-order conditions have a single degree of freedom.

We first show that the initial conditions in the $n \times 2$ case have a single degree of freedom as well, despite the fact that $b_1(1) > b_2(1)$ (see Corollary 2). When $b_1(1) > b_2(1)$, there are two distinct intervals of bids between which the first-order conditions for the optimal bids change. Bids $b \in [0, b_2(1)]$ compete against first- and second-unit bids made by opponents. Let $F_1(v) \equiv F(v)^2$, $H_1(b) \equiv F_1(\varphi_1(b))$, $F_2(v) \equiv 2F(v) - F(v)^2$ and $H_2(b) \equiv F_2(\varphi_2(b))$. The first-order conditions for the first- and second-unit bids in this range simplify to

$$\left(\frac{h_2(b)}{H_2(b) - H_1(b)} + \frac{(n-2)h_1(b)}{H_1(b)}\right)(v_1 - b) = 1 \tag{7}$$

$$\frac{(n-1)h_1(b)}{2H_1(b)}(v_2 - b) = 1 \tag{8}$$

For bids $b \in (b_2(1), b_1(1)]$, the opposing bids are solely for the first unit changing the win probabilities for a bidder's first unit. The corresponding first-order condition for the first unit is

$$\frac{(n-2)h_1(b)}{H_1(b)}(v_1 - b) = 1 \tag{9}$$

Since only first-unit bids are submitted in $(b_2(1), b_1(1)]$ and assuming that bidders use symmetric strategies, the bid function that solves (9) can be represented explicitly up to an unknown bid, because it reduces to the ODE one would get from a symmetric first-price auction with $n-1$ total bidders.

If $b_1(1)$ is known, the unique solution to this ODE can be represented as

$$b_1(1) - b_1(v_1)F(v_1)^{n-2} = \int_{v_1}^{1} x\, dF(x)^{n-2}. \tag{10}$$

To reiterate, this is an explicit characterization of first-unit bids on the interval $(b_2(1), b_1(1)]$ for a known value of $b_1(1)$. We next argue that the value of $b_2(1)$ is pinned down in equilibrium by $b_1(1)$. Note that the first-order condition for the first-unit in (9) is the same as the one in (7) with $h_2(b) = 0$. We also observe that it must be that $h_2(b_2(1)) = 0$ because the density of a second-order statistic vanishes at the upper bound of its support. The inverse bid function associated with the solution in (10) therefore satisfies (7) in a neighborhood of $b_2(1)$. This implies that a necessary condition for the selection of $b_2(1)$ is that it be optimally chosen according to (8) where $H_1(b)$ is the bid distribution determined by the initial choice of $b_1(1)$ and (10). In other words, at $b_2(1)$ the values of h_1 and H_1 are known up to $b_1(1)$.

In the uniqueness argument given in Section 5.2, the second step is to show that given two distinct initial conditions for the ODE derived from two distinct choices for the common high bid, the corresponding solutions to the ODE are monotonic in these initial conditions at all points in the interior of the domain. The same property holds in the $n \times 2$, which we record as Lemma 4. Similar to before, we view (7) and (8) as an ODE in unknown H_1 and H_2 with domain $(0, b_2(1)]$ and initial conditions determined by the value taken by $b_2(1)$ and the value of $\bar{v}_1 \equiv \varphi_1(b_2(1))$, where φ_1 is determined at $b_2(1)$ by (10). Using (7) and (8) express this ODE as

$$h_2(b) = \frac{H_2(b) - H_1(b)}{\varphi_1(b) - b} - \frac{2(n-2)}{n-1}\frac{H_2(b) - H_1(b)}{\varphi_2(b) - b} \tag{11}$$

$$h_1(b) = \frac{2}{n-1}\frac{H_1(b)}{\varphi_2(b) - b}, \tag{12}$$

where $\varphi_k(b) \equiv F_k^{-1}(H_k(b))$.

Lemma 4. *Let $\hat{b}_1(1) < \bar{b}_1(1)$ be two initial choices for $b_1(1)$ and $\hat{b}_2(1) < \bar{b}_2(1)$ be the corresponding choices for $b_2(1)$. Let $\hat{H}_1$ and $\hat{H}_2$ solve* (11) *and* (12) *when the initial condition is $\hat{H}_2(\hat{b}_2(1)) = 1$ and $\hat{H}_1(\hat{b}_2(1)) = F_1(\hat{v}_1)$, and let $\bar{H}_1$ and $\bar{H}_2$ solve* (11) *and* (12) *when the initial condition is $\bar{H}_2(\bar{b}_2(1)) = 1$ and $\bar{H}_1(\bar{b}_2(1)) = F_1(\bar{v}_1)$. For all $b \in (0, \hat{b}_2(1))$, $\hat{H}_1(b) > \bar{H}_1(b)$ and $\hat{H}_2(b) > \bar{H}_2(b)$.*

Proof. Since the equilibrium bid functions are increasing we have $\hat{H}_k(\hat{b}_2(1)) = 1 > \bar{H}_2(\hat{b}_2(1))$ for $k = 1, 2$. We show next that this inequality holds for all bids in $(0, \hat{b}_2(1)]$. To do this, we rule out that $\hat{H}_k$ crosses $\bar{H}_k$ for either k at any point in the domain. Let $\hat{b} < b_2(1)$ represent the largest bid at with either $\hat{H}_1$ crosses $\bar{H}_1$ or $\hat{H}_2$ crosses $\bar{H}_2$. Consider first the case where $\hat{H}_1(\hat{b}) = \bar{H}_1(\hat{b})$ and $\hat{H}_2(\hat{b}) > \bar{H}_2(\hat{b})$. Using (12), this implies that $\hat{h}_1(\hat{b}) < \bar{h}_1(\hat{b})$, but this implies that $\bar{H}_1$ crosses $\hat{H}_1$ from above, a contradiction. Similarly, one can show using (11) that $\hat{H}_1(\hat{b}) > \bar{H}_1(\hat{b})$ and $\hat{H}_2(\hat{b}) = \bar{H}_2(\hat{b})$ implies $\hat{h}_2(\hat{b}) < \bar{h}_2(\hat{b})$. If it were true that $\hat{H}_1(\hat{b}) = \bar{H}_1(\hat{b})$ and $\hat{H}_2(\hat{b}) = \bar{H}_2(\hat{b})$ (i.e., they both crossed together), then we would have to conclude by the FTODE that $\hat{b}_2(1) = \bar{b}_2(1)$, because the FTODE implies that there is a unique solution to the system in (11) and (12) beginning from an initial value at a $\hat{b} \in (0, \hat{b}_2(1)]$. $\square$

Lemma 4 implies that any two solutions to (11) and (12) are ordered pointwise in the interior of the domain according to the ordering of the high bids on the first unit. The final step in the uniqueness proof is to use this fact to rule out that there can be more than one valid choice of $b_1(1)$.

In the literature on uniqueness in first-price auctions, an additional assumption is required to complete this final step. This may be an assumption that L'Hopital's rule can be applied to the ODE at the low bid;[25] an assumption that there is a binding reserve price or an atom at the lower end of the support of values [Lebrun, 1999, Maskin and Riley, 2003]; or an assumption about the properties of the value distribution in an interval including the lower bound of the support [Lebrun, 2006]. Each of these approaches apply here as well, using the implications of Lemma 4 and the equation in (12).

The equation in (12) implies that for two bids, $b < b'$,

$$\frac{H_1(b')}{H_1(b)} = \exp\left\{ \frac{2}{n-1} \int_b^{b'} \frac{dx}{\varphi_2(x) - x} \right\}. \tag{13}$$

As in Lemma 4, let $\hat{b}_2(1) < \bar{b}_2(1)$ so that $\hat{H}_1(b) > \bar{H}_1(b)$ and $\hat{H}_2(b) > \bar{H}_2(b)$ for all $b \in (0, \hat{b}_2(1))$. From (13) and Lemma 4, it follows that

$$1 < \frac{\hat{H}_1(b')}{\bar{H}_1(b')} < \frac{\hat{H}_1(b)}{\bar{H}_1(b)}. \tag{14}$$

With an atom at the bottom of the distribution F equal to c it must be in equilibrium that $H_1(0) = c^2$, implying that if $\hat{H}_1$ and $\bar{H}_1$ both derive from equilibrium strategies $\hat{H}_1(0)/\bar{H}_1(0) = 1$. But this requirement conflicts with (14) which bounds this ratio away from one for all $b < b'$. We conclude that $\hat{H}_1$ and $\bar{H}_1$ cannot both derive from equilibrium strategies.

[25] Lebrun [2006] points out that this assumption is implicit in Bajari [2001].

Proposition 6. *In the last-accepted-bid auction with n symmetric bidders each with demand for the two available goods determined by two independent draws from the distribution F where $F(0) > 0$, there is a unique symmetric equilibrium with differentiable bid functions satisfying* (11) *and* (12)*.*

The restriction in Proposition 6 to choices of F with an atom at the lower endpoint can be replaced with other another assumption such as an assumption on the validity of using L'Hopital's rule at the lower endpoint as in Bajari [2001]. However, the discussion in Lebrun [2006] suggests that *some* additional assumption about the bidding behavior at the lower endpoint is needed to prove uniqueness in the first-price auction. Given the close relation of our model to the first-price uniqueness problem, we do not believe that we can prove uniqueness without an additional assumption. We use an atom at the bottom of F, which may arise from the use of a reserve price and may be arbitrarily small, because it yields the simplest uniqueness argument while still showing a close connection with the corresponding first-price problem.

6 Properties of Other Auction Equilibria

In this section, we contrast the properties of the LAB section discussed in the previous sections with those of two other common multi-unit pricing rules, the pay-as-bid (PAB) auction and the first-rejected-bid (FRB) uniform-price auction. Specifically, we focus on issues surrounding separation and uniqueness of equilibrium.

All of the auctions we analyze have a structure in which bids for different units are co-determined only when the monotonicity constraint (that bids must be weakly decreasing in quantity) is binding.[26] We capture this structure with the notion of separable incentives.

Definition 5 (Separable incentives)**.** *An auction model has* separable incentives *if there are functions $((u_k^i)_{k=1}^{m_i})_{i=1}^n$ such that for each agent i, each unit k, and all bid profiles (b^i, b^{-i}) and value profiles v,*

$$u^i\left(b^i, b^{-i}; v\right) = \sum_{k=1}^{m_i} u_k^i\left(b_k^i, b^{-i}; v_k\right).$$

[26] Separately, these models exhibit the standard IPV mechanism design monotonicity-in-value. Because this is a result and not a constraint, when we refer to binding monotonicity constraints we are referring to monotonicity in quantity.

When quantity-monotonicity constraints are not binding, a model with separable incentives can be analyzed dimension-by-dimension as a set of m independent optimization problems. In light of Lemma 5 this features prominently in our analysis of the revelation properties of the FRB and PAB auction formats.

Lemma 5 (Separability of multi-unit auctions)**.** *The FRB, LAB, and PAB auctions each have separable incentives. In each of these auctions, each dimensional utility function u_k^i satisfies increasing differences in (b_k, v_k).*

Lemma 5 is proved in Appendix B. If the quantity-monotonicity constraint does not bind, in models with separable incentives the bid for any unit is determined solely by the value for this unit and the opponent's bidding strategy. Thus in a separating equilibrium, separable incentives imply that the agent's optimization problem is well-behaved and independent of any constraints.

Definition 6 (Partial pooling)**.** *A bid function b^i exhibits* partial pooling *if the inverse bid correspondence is multi-valued with positive probability; that is,*

$$\Pr\left(v \in \left\{v' : \#\varphi^i\left(b^i\left(v'\right)\right) > 1\right\}\right) > 0.$$

There is a wedge between strict separation (Definition 2) and partial pooling: inverse bids might be multi-valued with zero probability. We are concerned with issues of information confounding, and in particular in situations in which information is obfuscated in equilibrium. If equilibrium bids are non-separating with probability zero, equilibrium is essentially separating, and this distinction is not meaningful.

Our definition of partial pooling is structured to capture two separate pooling effects. In the FRB auction, truthful bidding for the first unit is a weakly dominant strategy. However, we show that there is a range of last-unit valuations such that a bid of zero strictly dominates all others; this occurs because residual competition comes from opponents' low units, for which distributions are relatively strong. Increasing the bid for the final unit has little marginal effect on the probability of winning the unit but a comparatively strong marginal effect on the expected cost paid for all $m_i - 1$ units, conditional on their being won. In an equilibrium with truthful bids for the first unit, the probability of witnessing any particular bid profile is zero even though the probability of witnessing a zero bid for an agent's final unit is strictly positive; partial pooling captures this positive-probability noninvertibility.

Partial pooling also captures the information confounding we observe in the PAB auction. In PAB, the bidder is facing increasingly aggressive competition as she considers her bid for higher units: her bid for higher units is against her opponents' bids for lower units. We show that there is generally an incentive for the idealized bid for the first unit to be below the idealized bid for the second unit, violating the bid monotonicity constraint. This implies that, for certain value profiles, bids will be flat for small quantities. Continuity of utility in value implies that this same flat will be realized for nearby value profiles—if, for example, the value for the first unit falls while the value for the second unit rises (or vice versa)—and thus upon witnessing a particular flat bid the bidder's value profile cannot be perfectly inverted. Again, this happens in spite of no bid being submitted with positive probability.

Aside from implications for tractability, information revelation is directly related to efficiency. An efficient mechanism must allocate units to the agents with the highest values. When information is confounded, this is not possible: efficiency entails knowing which agents have the highest values for the m available units, and standard identification arguments imply that if this is possible, bids must be separating. We thus contrast the FRB and PAB auctions, in which all equilibria exhibit partial pooling and are thus inefficient, with the LAB auction, which we have shown to admit a separable and efficient equilibrium without pooling.

Remark 1. *Any pure-strategy equilibrium can be transformed into a monotone pure-strategy equilibrium without affecting agents' incentives or payoffs. We therefore restrict attention to equilibria in monotone pure strategies.*

Lemma 6 (Separable bids in separating equilibrium)**.** *In a monotone strictly separating equilibrium the LAB, FRB, and PAB auction models, bidder i's equilibrium bid function can be written as*

$$b^i(v) = \left(b_1^i(v_1), \ldots, b_{m_i}^i(v_{m_i})\right).$$

Corollary 3 (No mass points in separating equilibrium)**.** *In a monotone strictly separating equilibrium, there is no bidder i, unit k, and nondegenerate interval $(\underline{v}, \overline{v})$ such that $b_k^i|_{(\underline{v},\overline{v})}$ is constant.*

Taken together, the above results imply that either equilibrium bids can be anlyzed independently, unit-by-unit, or equilibrium exhibits partial pooling. Following the definition of partial pooling, this implies that when bids cannot be analyzed independently equilibrium outcomes must be inefficient, and information is not fully revealed. Helpfully these results allow us to analyze the revelation question dimension-by-dimension and, from these dimensional analyses, to build contradictions which expose the relevance of partial pooling.

6.1 Partial Pooling in the First-Rejected-Bid Auction

In the FRB auction bids for large quantities are disproporitionately unprofitable. A small increase in bid for a large quantity implies that, when this bid is supra-marginal, this increase is paid for each unit won. Because a bidder competes for large quantities against her opponents' small quantities, not only is there an outsized cost associated with increasing this bid but there is also only a relatively small gain. These incentives balance in favor of a mass point at a bid of zero.

To eliminate pathological cases, we define the notion of a well-behaved equilibrium.

Definition 7. *A bid function b_k^i is* well-behaved *if $d_+^t b_k^i / dv^t$ is bounded on $(0,1)$, for all finite t. The bid profile (b^i) is well-behaved if b_k^i is well-behaved for all agents i and all units k.*

Lemma 7 (Partial pooling in FRB)**.** *All well-behaved equilibria of the FRB auction with $m \geq 2$ units exhibit partial pooling.*

Proof. The unitwise utility function in the FRB auction can be expressed as

$$u_k^i\left(b_k^i, b^{-i}; v_k\right) = \left(v_k - b_k^i\right) H_{m-k+1}^{-i}\left(b_k^i\right) \\ - (k-1)\int_0^{b_k^i} H_{m-k+2}^{-i}(x) - H_{m-k+1}^{-i}(x)\,dx.$$

Since for any unit the agent has the option of bidding 0 and obtaining (at worst) zero utility, $u_k^i(b_k^i(v), b^{-i}; v_k) \geq 0$ whenever b_k^i is a best response bidding function. As established above, if equilibrium does not exhibit partial pooling, $b_k^i(v) \equiv b_k^i(v_k)$. It follows that in an equilibrium without partial pooling,

$$\frac{v_k - b_k^i\left(v_k\right)}{k-1} \geq \frac{\int_0^{b_k^i(v_k)} H_{m-k+2}^{-i}(x) - H_{m-k+1}^{-i}(x)\,dx}{H_{m-k+1}^{-i}\left(b_k^i\left(v_k\right)\right)}.$$

Strict separation, well-behavedness, and best-responsiveness require that $b_k^i(v_k) > 0$ whenever $v_k > 0$, that bids are dense near 0, and that $b_k^i(0) = 0$.[27] Then in the limit, for all $k > 1$,

$$\lim_{b \searrow 0} \frac{\int_0^b H_{m-k+2}^{-i}(x) - H_{m-k+1}^{-i}(x)\, dx}{H_{m-k+1}^{-i}(b)} = 0.^{28}$$

When equilibrium is well-behaved and arbitrarily differentiable, for a set of relevant $t \in \{0, 1, \ldots, \bar{t}\}$ l'Hôpital's rule implies

$$\lim_{b \searrow 0} \frac{d^{(t)} H_{m-k+2}^{-i}(b) - d^{(t)} H_{m-k+1}^{-i}(b)}{d^{(t+1)} H_{m-k+1}^{-i}(b)} = 0.^{29}$$

Quantity-monotonicity requires that $b_{k'}^j(v) > b_{k'+1}^j(v)$, and hence by the nature of the order-statistic model there is some t such that

$$\lim_{b \searrow 0} \left| d^{(t)} H_{m-k+2}^{-i}(b) \right| > \lim_{b \searrow 0} \left| d^{(t)} H_{m-k+1}^{-i}(b) \right| = 0.$$

At this t, well-behavedness requires that $\lim_{b \searrow 0} |d^{(t+1)} H_{m-k+1}^{-i}(b)| \geq 0$ is finite, hence the limit is strictly positive, contradicting strict separation. $\square$

It is worth clarifying the role of well-behavedness in Lemma 7. If the necessary limit (in the proof) does not exist, the Lemma is automatically satisfied: the limit will fail to exist only when the ratio can be discretely positive for b arbitrarily close to 0. This alone is sufficient to indicate that pooling at 0 is advantageous. Thus well-behavedness supports bid density, and provides that $d^{(t+1)} H_{m-k+1}^{-i}(0)$ is finite at the smallest t for which it is nonzero. We do not have a clean economic interpretation for what it would mean for this derivative to be infinite while all lower derivatives are zero, but nor can we rule it out out of hand.

[27] In a working version of this paper we provide arguments that these statements continue to hold in the presence of a reserve price $r > 0$.

[28] Technically only a weak inequality, ≤ 0, is required. Given the relationship between H_{m-k+2}^{-i} and H_{m-k+1}^{-i} it is straightforward to show that strict inequality cannot be satisfied.

[29] This limit makes clear the hidden role of the assumption that $m \geq 2$ units are available. When $m = 1$, $H_{m-k+2}^{-i} = 0$, invalidating this proof approach. This is to be expected, since with $m = 1$ unit available the FRB auction is equivalent to a second-price auction, which admits a well-behaved, separating, truthful equilibrium.

Remark 2. *With $n = 2$ bidders and $m \geq 2$ units, all equilibria of the FRB auction in weakly-dominant strategies exhibit partial pooling. Because truthful reporting for the first unit is weakly-dominant, $d^{(t)}H_1^{-i}(0)$ is finite for the lowest t at which it is nonzero, implying that Lemma 7 can be applied directly.*

Remark 2 makes use of a further wrinkle in well-behavedness. It is not necessary that all H_{m-k+1}^{-i} be well-behaved, only that there exists an agent i and a unit k such that H_{m-k+1}^{-i} is well-behaved. This allows for the following proposition.

Proposition 7 (Inefficient equilibrium in FRB)**.** *All equilibria of the FRB auction satisfy one (or more) of the following two properties:*

i. Equilibrium is inefficient.

ii. For all agents i and all units k, $d^{(t)}H_{m-k+1}^{-i}(0)$ is infinite at the lowest t at which it is nonzero.

6.2 Partial Pooling in the Pay-as-Bid Auction

As the bid-for quantity increases, the marginal distribution of opponent values shifts upward; the relative lack of competition for small quantities implies partial pooling in the PAB auction. In the case of two bidders a bidder will win unit 1 if and only if her opponent does not win unit m; similarly the bidder will win unit 2 if and only if her opponent does not win unit $m-1$. Since her opponent's marginal distribution of values for unit $m-1$ dominates the distribution of values for unit m, the bidder faces less competition for unit 1 than she does for unit 2. Ideally, given a value $v_1 = v, v_2 = v$ she would bid less for unit 1 than for unit 2, bumping into the quantity-monotonicity constraint.[30]

When the quantity-monotonicity constraint is binding, bids cannot be written as products of independent bids across the units the bidder demands. Intuition suggests that partial pooling is present: if the bid $b_1 = b, b_2 = b$ is observed, it cannot be known whether this bid has resulted from individual unconstrained bids, or bids that pass through the monotonicity constraint. Then values cannot be inverted out from observed bids, and there is some degree of pooling present in equilibrium.

[30]This was explored in a divisible-good context by Woodward [2016].

Lemma 8 (Equal upper bounds)**.** *Let $\bar{b}$ be the maximum bid submitted for any unit, and let $\bar{b}_k^i$ be bidder i's maximum bid for unit k. There are two bidders, i and j, such that $\bar{b}_k^i = \bar{b}_{k'}^j$ for all units k and k'.*

We now turn attention to *continuous-bid* equilibria, in which bids are continuous functions of value. Continuity is the barest form of equilibrium tractability. In the two-bidder case standard aguments suffice to rule out discontinuities in equilibrium bid functions, but in general this is less clear.[31] Nonetheless, we are able to show that equilibria in continuous bids exhibit partial pooling. Since partial pooling implies inefficiency, as do discontinuous bids, it follows that all equilibria in the PAB auction are inefficient. Further, inasmuch as partial pooling makes equilibrium computation more difficult, and discontinuous equilibria also present computational challenges, these results can be taken as suggesting a general intractability of PAB auction equilibria.[32]

We define a *maximal* bidder as a bidder i such that $\bar{b}_k^i = \bar{b}$ for all k; Lemma 8 implies that at least two maximal bidders exist.

Lemma 9 (Monotonicity of distributional differences)**.** *Let i be a maximal bidder, and let $\delta_k(b) = H_{m-k+1}^{-i}(b) - H_{m-k}^{-i}(b)$. In any strictly-separating, continuous-bid equilibrium of the PAB auction, $\lim_{b \nearrow \bar{b}} \delta_k(b) > 0$ for all $k \in \{1, \ldots, m-1\}$.*

Corollary 4 (Partial pooling in PAB)**.** *Contiunous-bid equilibria in the pay-as-bid auction exhibit partial pooling.*

Remark 3. *With $n = 2$ bidders, all equilibria of the PAB auction exhibit partial pooling. When there are only two bidders in the model, the problem is analogous to a set of simultaneous two-bidder asymmetric auctions. Since the support of valuations is convex, standard results imply that bids are continuous in value, satisfying the antecedent of Lemma 4.*

Proposition 8 (Inefficient equilibrium in PAB)**.** *All equilibria of the pay-as-bid auction are inefficient.*

[31] This is observed throughout the single-unit auction literature; with multiple units, the problem is exacerbated. For example, if each bidder has positive value for all m units ($m_i = m$ for all i), then when determining the bid for her $m - 2^{\text{nd}}$ unit a bidder must consider not only the possibility that any of her opponents receives 2 units and all the others receive 0 units (this is analogous to the single-unit case), but also to the possibility that any combination of two opponents each receives 1 unit while all others receive 0 units. This iso-allocation set makes standard no-gaps arguments inapplicable.

[32] This is in line with known results about multi-unit auctions, including Hortaçsu and Kastl [2012].

Proof. In equilibrium, bid functions are either continuous or discontinuous. If they are continuous, equilibrium exhibits partial pooling (Lemma 4). Because the closure of the equilibrium bid set must be convex, when bids are discontinuous it must be that higher-value agents sometimes lose a unit to lower-value agents, implying inefficiency. In either case, outcomes are inefficient. □

6.3 Low-Revenue Equilibria and Multiplicity

It has been noted (e.g., Engelbrecht-Wiggans and Kahn [1998], and Ausubel et al. [2014]) that the FRB auction frequently admits equilibria with arbitrarily small seller revenue. As implied by the proof of Lemma 7, all equilibria in the FRB auction yield zero revenue with positive probability. We show now that the LAB auction does not admit any such equilibria.

Proposition 9 (Zero-revenue equilibrium in FRB)**.** *The FRB auction always admits zero-revenue equilibria. The LAB auction never admits zero-revenue equilibria.*

Proof. Construction of low-revenue equilibria in a multi-unit setting is similar to that in a single-unit setting:[33] take numbers $(\tilde{m}_i) \in \mathbb{N}_0^n$ such that $\sum_{i=1}^n \tilde{m}_i = m$. Pick $\bar{s} \geq 1$ and let bidder i submit the bid

$$b^i(q; v) = \begin{cases} \bar{s} & \text{if } q \leq \tilde{m}_i, \\ 0 & \text{otherwise.} \end{cases}$$

Then the equilibrium price is always zero, independent of the agents' private information; moreover, to win a greater quantity agent i must bid $\bar{s}$ for unit $\tilde{m}_i + 1$, obtaining weakly negative gross utility on this unit and incurring an additional payment of $\tilde{m}_i \bar{s}$. This is never utility-improving, hence these bid functions represent an equilibrium.

It is straightforward to show, by contradiction, that the LAB auction does not admit zero-revenue equilibria. Letting $q^i(v^i, v^{-i})$ be the equilibrium quantity allocation of agent i given value profiles v^i and v^{-i}, it is without loss of generality to assume that $q^i(v^i, v^{-i}) < m$ with positive probability.[34] Note that for almost all v^{-i} such that $q^i(v^i, v^{-i}) < m$, $b^i(q^i(v^i, v^{-i}) + 1; s_i) = 0$; furthermore, $b^{-i}(m - q^i(v^i, v^{-i}) + 1; v^{-i}) = 0$. Then by increasing her bid for units $q > q^i(v^i, v^{-i})$ to $\varepsilon > 0$, bidder i will incur an additional cost of at most

[33] See, e.g., Milgrom [2004], pages 262–264.

[34] This follows from market clearing and the fact that we can focus on any particular agent.

$m\varepsilon$ but will win unit $q^i(v^i, v^{-i}) + 1$ with discretely positive probability. For ε sufficiently small this deviation is profitable, hence there is no low-revenue equilibrium. $\square$

Proposition 9 establishes that the FRB pricing rule admits many zero-revenue equilibria, however the notion of "many" in this context is ill-defined. In particular, it is possible that the zero-revenue equilibria have measure zero in the set of all equilibria. This caveat aside, because the zero-revenue equilibria are simple focal points for collusive behavior, whether they are measurably present in the set of all equilibria does not affect the fact that they present a real practical concern.

7 Conclusion

We have defined a model of multi-unit auctions in which bidders have private values given by ordered draws from a single distribution. In this model, we show that the last accepted bid uniform-pricing rule induces bidding incentives analogous to those in a single-unit first-price auction. We show that the last accepted bid auction can admit a tractable representation and can be both efficient and fully-revealing of bidders' private information. By noting the connection between bidding incentives in a single-unit first-price auction and a multi-unit last accepted bid auction, we identify a new salient feature common to both auctions: in both auctions, bidders pay the highest market-clearing price.

We compare the last accepted bid auction to the first rejected bid uniform-price and the pay-as-bid auctions. We show that in each of these auctions bidder information is confounded in all well-behaved equilibria. This further implies that these auctions are generally inefficient. We provide an additional construction which emphasizes that the first rejected bid auction always admits low-revenue equilibria, a phenomenon which cannot be sustained in the last accepted bid auction.

Taken as a whole, our results are strongly in favor of employing the last accepted bid pricing rule rather than the first rejected bid pricing rule when a uniform-price auction is implemented. To date the literature has overlooked the possibility of a meaningful difference between the two; we show that this difference is real and has material implications in support of the last accepted bid auction.

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A First-order conditions for leading example

A.1 Last accepted bid

The agent's utility can be expressed as

$$\begin{aligned} u^i\left(b^i; v^i\right) = {} & v_1^i F^{(2)} \circ \varphi_2^{-i}\left(b_1^i\right) + v_2^i F^{(1)} \circ \varphi_1^{-i}\left(b_2^i\right) \\ & - \left(F^{(2)} \circ \varphi_2^{-i}\left(b_1^i\right) - F^{(1)} \circ \varphi_1^{-i}\left(b_1^i\right)\right) b_1^i - 2 b_2^i F^{(1)} \circ \varphi_1^{-i}\left(b_2^i\right) \\ & - \int_{\varphi_1^{-i}\left(b_2^i\right)}^{\varphi_1^{-i}\left(b_1^i\right)} b_1^{-i}\left(v\right) dF^{(1)}\left(v\right). \end{aligned}$$

From here, it is straightforward to compute the model's first-order conditions,

$$\begin{aligned} \frac{\partial}{\partial b_1^i} &: \left(v_1^i - b_1^i\right) dF^{(2)} \circ \varphi_2^{-i}\left(b_1^i\right) d\varphi_2^{-i}\left(b_1^i\right) - \left(F^{(2)} \circ \varphi_2^{-i}\left(b_1^i\right) - F^{(1)} \circ \varphi_1^{-i}\left(b_1^i\right)\right); \\ \frac{\partial}{\partial b_2^i} &: \left(v_2^i - b_2^i\right) dF^{(1)} \circ \varphi_1^{-i}\left(b_2^i\right) d\varphi_1^{-i}\left(b_2^i\right) - 2F^{(1)} \circ \varphi_1^{-i}\left(b_2^i\right). \end{aligned}$$

Assuming a symmetric equilibrium many subscripts can be dropped; substituting in for the known order statistic distributions ($F^{(1)}(x) = x^2$, $F^{(2)}(x) = 2x - x^2$) gives

$$2\left(\varphi_1(b) - b\right)\left(1 - \varphi_2(b)\right) d\varphi_2(b) - \left(2\varphi_2(b) - \varphi_2(b)^2 - \varphi_1(b)^2\right) = 0;$$
$$\left(\varphi_2(b) - b\right)\varphi_1(b)\, d\varphi_1(b) - \varphi_1(b)^2 = 0.$$

A.2 First rejected bid

The agent's utility can be expressed as

$$\begin{aligned} u^i\left(b^i; v^i\right) = {} & v_2^i F^{(1)} \circ \varphi_1^{-i}\left(b_2^i\right) - 2\int_0^{\varphi_1^{-i}\left(b_2^i\right)} b_1^{-i}(v)\, dF^{(1)}(v) \\ & - \left(F^{(2)} \circ \varphi_2^{-i}\left(b_2^i\right) - F^{(1)} \circ \varphi_1^{-i}\left(b_2^i\right)\right) b_2^i \\ & + v_1 F^{(2)} \circ \varphi_2^{-i}\left(b_1\right) - \int_{\varphi_2^{-i}\left(b_2^i\right)}^{\varphi_2^{-i}\left(b_1^i\right)} b_2^{-i}(v)\, dF^{(2)}(v). \end{aligned}$$

From here, it is straightforward to compute the model's first-order conditions,

$$\frac{\partial}{\partial b_1^i} : \left(v_1^i - b_1^i\right) dF^{(2)} \circ \varphi_2^{-i}\left(b_1^i\right) d\varphi_2^{-i}\left(b_1^i\right);$$
$$\frac{\partial}{\partial b_2^i} : \left(v_2^i - b_2^i\right) dF^{(1)} \circ \varphi_1^{-i}\left(b_2^i\right) d\varphi_1^{-i}\left(b_2^i\right) - \left(F^{(2)} \circ \varphi_2^{-i}\left(b_2^i\right) - F^{(1)} \circ \varphi_1^{-i}\left(b_2^i\right)\right).$$

B Proofs of information pooling properties

Proof of Lemma 5. We analyze each auction in turn. Note that it is without loss in each case to consider the agent as bidding for all m available units, with the constraint that she has zero value for units $k > m_i$.

FRB. Utility is written as

$$\begin{aligned} u^i\left(b^i, b^{-i}; v\right) = {} & \sum_{k=1}^{m}\left(\sum_{k'=1}^{k} v_{k'} - k b_{k+1}^i\right) \Pr\left(b_{m-k}^{-i} \geq b_{k+1}^i \geq b_{m-k+1}^{-i}\right) \\ & + \left(\sum_{k'=1}^{k} v_{k'} - k\mathbb{E}\left[b_{m-k+1}^{-i} \middle| b_k^i \geq b_{m-k+1}^{-i} \geq b_{k+1}^i\right]\right) \\ & \times \Pr\left(b_k^i \geq b_{m-k+1}^{-i} \geq b_{k+1}^i\right). \end{aligned}$$

The relevant probabilities are

$$\begin{aligned}
\Pr\left(b^{-i}_{m-k} \geq b^i_{k+1} \geq b^{-i}_{m-k+1}\right) &= H^{-i}_{m-k+1}\left(b^i_{k+1}\right) - H^{-i}_{m-k}\left(b^i_{k+1}\right), \\
\Pr\left(b^i_k \geq b^{-i}_{m-k+1} \geq b^i_{k+1}\right) &= H^{-i}_{m-k+1}\left(b^i_k\right) - H^{-i}_{m-k+1}\left(b^i_{k+1}\right).
\end{aligned}$$

Algebraic manipulation gives a separable utility form of

$$\begin{aligned}
u^i\left(b^i, b^{-i}; v\right) = \sum_{k=1}^{m} & v_k H^{-i}_{m-k+1}\left(b^i_k\right) - \left(H^{-i}_{m-k+2}\left(b^i_k\right) - H^{-i}_{m-k+1}\left(b^i_k\right)\right)(k-1)\, b^i_k \\
& - k\int_0^{b^i_k} b\, dH^{-i}_{m-k+1} + (k-1)\int_0^{b^i_k} b\, dH^{-i}_{m-k+2}.^{35}
\end{aligned}$$

LAB. Utility is written as

$$\begin{aligned}
u^i\left(b^i, b^{-i}; v\right) = \sum_{k=1}^{m} & \left(\sum_{k'=1}^{k} v_{k'} - k b^i_k\right) \Pr\left(b^{-i}_{m-k} \geq b^i_k \geq b^{-i}_{m-k+1}\right) \\
& + \left(\sum_{k'=1}^{k} v_{k'} - k\mathbb{E}\left[b^{-i}_{m-k} \middle| b^i_k \geq b^{-i}_{m-k} \geq b^i_{k+1}\right]\right) \\
& \times \Pr\left(b^i_k \geq b^{-i}_{m-k} \geq b^i_{k+1}\right).
\end{aligned}$$

The relevant probabilities are

$$\begin{aligned}
\Pr\left(b^{-i}_{m-k} \geq b^i_k \geq b^{-i}_{m-k+1}\right) &= H^{-i}_{m-k+1}\left(b^i_k\right) - H^{-i}_{m-k}\left(b^i_k\right), \\
\Pr\left(b^i_k \geq b^{-i}_{m-k} \geq b^i_{k+1}\right) &= H^{-i}_{m-k}\left(b^i_k\right) - H^{-i}_{m-k}\left(b^i_{k+1}\right).
\end{aligned}$$

Algebraic manipulation gives a separable utility form of

$$\begin{aligned}
u^i\left(b^i, b^{-i}; v\right) = \sum_{k=1}^{m} & v_k H^{-i}_{m-k+1}\left(b^i_k\right) - \left(H^{-i}_{m-k+1}\left(b^i_k\right) - H^{-i}_{m-k}\left(b^i_k\right)\right) k b^i_k \\
& - k\int_0^{b^i_k} b\, dH^{-i}_{m-k} + (k-1)\int_0^{b^i_k} b\, dH^{-i}_{m-k+1}.
\end{aligned}$$

[35] This form is a convenient symmetric shorthand, but H^{-i}_{m+1} is ill-defined. Since this term is in $[0,1]$ and is always premultiplied by $(1-1)=0$, the exact specification is irrelevant.

PAB. This is essentially trivial. Note that utility has a naturally separable form,

$$u^i\left(b^i, b^{-i}; v\right) = \sum_{k=1}^{m_i} \left(v_k - b^i_k\right) H^{-i}_{m-k+1}\left(b^i_k\right).$$

□

Proof of Lemma 6. Note that strict separation implies that strategies are strictly monotone in value. If strategies cannot be separated as in the statement of the Lemma, the monotonicity constraint must be binding.[36]

Suppose first that bids are continuous in value. If the bid profile cannot be written as a product of independent dimensional bids, the monotonicity constraint must be binding over some product of nondegenerate intervals $[\underline{b}_k, \bar{b}_k) \times \cdot \times [\underline{b}_{k'}, \bar{b}_{k'})$. Because bids are continuous in value, bids are not invertible on this range; since this range has positive measure (and can be expanded to account for higher and lower units for which the monotonicity constraints are not binding) it follows that equilibrium is not strictly separating, a contradiction.

Now suppose that the monotonicity constraint is binding at a point at which bids are discontinuous in value. Because dimensional utilities satisfy increasing differences and are continuous in value, there is a neighborhood below this point on which the monotonicity constraint is binding and bids are locally continuous. Then the above argument holds. □

Proof of Lemma 8. Let $\bar{b}^i_k$ be bidder i's maximum bid for unit k; without loss of generality, this is $\bar{b}^i_k = b^i_k(\bar{v})$. Suppose that $\bar{b}^i_k \neq \bar{b}^{-i}_{m-k+1}$, and without loss of generality assume that $\bar{b}^i_k > \bar{b}^{-i}_{m-k+1}$. Then anytime bidder i submits a bid $b \in (\bar{b}^{-i}_{m-k+1}, \bar{b}^i_k]$, she wins unit k with probability 1; she could reduce her bid without affecting her winning probability, improving her utility. Then $\bar{b}^i_k = \bar{b}^{-i}_{m-k+1}$ for all units k.

Bid monotonicity requires that $\bar{b}^i_k \geq \bar{b}^i_{k'}$ for all $k' \geq k$. Then

$$\bar{b}^i_k \geq \bar{b}^i_{k'} = \bar{b}^{-i}_{m-k'+1} \geq \bar{b}^{-i}_{m-k+1} = \bar{b}^i_k.$$

Then $\bar{b}^i_k = \bar{b}^i_{k'}$ for all $k' \geq k$, and bidder i's maximum bid is independent of the unit she is bidding for. Since this maximum bid is equal to bidders' $-i$

[36]Due to standard peculiarities of measure zero, this statement is true when constrained to left-continuous bid functions but may fail in the presence of arbitrary discontinuities. It is straightforward to show that any monotone strictly separating equilibrium is incentive equivalent to an equilibrium which is left continuous in value, thus this statement is more or less without loss of generality.

maximum bid for the complementary unit, the maximum bid is independent of unit and agent. □

Proof of Lemma 9. If bidder i's best-response bid function for unit k, $b_k^\star$, is continuous, it must be that H_{m-k+1}^{-i} is continuous; moreover, where H_{m-k+1}^{-i} is not differentiable it has a "downward" kink. Following the first-order conditions of the model, for any $v \in (0,1)$ it must be that either

$$b_k^\star(v) > b_{k+1}^\star(v), \text{ or } d_+ H_{m-k}^{-i}\left(b_k^\star(v)\right) < d_+ H_{m-k+1}^{-i}\left(b_k^\star(v)\right) \text{ (or both).}^{37}$$

Since $b_{\tilde{k}}^\star$ is continuous for each $\tilde{k}$ (by assumption), whenever the first inequality holds it must hold over an interval. Note that is must be that there is some v for which the first inequality holds; otherwise $b_k^\star(v) = b_{k+1}^\star(v)$ for all v, implying the second inequality for all $b \in [0, \bar{b}]$. Then there is a mass point in H_{m-k}^{-i} at $\bar{b}$, which is inconsistent with i submitting a continuous bid function.[38]

Let v be such that $b_k^\star(v) > b_{k+1}^\star(v)$. Appealing to incentive compatibility and first-order dominance,

$$\begin{aligned}\left(v - b_k^\star(v)\right) H_{m-k+1}^{-i}\left(b_k^\star(v)\right) &\geq \left(v - b_{k+1}^\star(v)\right) H_{m-k+1}^{-i}\left(b_{k+1}^\star(v)\right) \\ &> \left(v - b_{k+1}^\star(v)\right) H_{m-k}^{-i}\left(b_{k+1}^\star(v)\right) \\ &\geq \left(v - b_k^\star(v)\right) H_{m-k}^{-i}\left(b_k^\star(v)\right).\end{aligned}$$

These inequalities imply

$$\frac{H_{m-k+1}^{-i}\left(b_k^\star(v)\right)}{H_{m-k}^{-i}\left(b_k^\star(v)\right)} > \frac{H_{m-k+1}^{-i}\left(b_{k+1}^\star(v)\right)}{H_{m-k}^{-i}\left(b_{k+1}^\star(v)\right)}.$$

Continuity and maximality imply that there is $\tilde{v}$ with $b_{k+1}^\star(\tilde{v}) = b_k^\star(v)$. Then

$$\frac{H_{m-k+1}^{-i}\left(b_k^\star(\tilde{v})\right)}{H_{m-k}^{-i}\left(b_k^\star(\tilde{v})\right)} \geq \frac{H_{m-k+1}^{-i}\left(b_{k+1}^\star(\tilde{v})\right)}{H_{m-k}^{-i}\left(b_{k+1}^\star(\tilde{v})\right)}.$$

[37] The use of $b_k^\star$ (instead of $b_{k+1}^\star$) is irrelevant here, and is used solely to ensure that attention is focused on a single bid. It is also sufficient to consider only right derivatives, which are finite at all relevant points (otherwise a slight increase in bid would trivially be profitable, and would be feasible since bids are necessarily below values whenever right derivatives are nonzero).

[38] Either i's bid function is discontinuous, or the high bid is such that $\bar{b} = 1$. In this latter case, the existence of a mass point implies some of i's opponents are bidding above their values and winning with positive probability, which is not a best response in the PAB auction.

Let $I(b)$ be the interval over which $b_k^\star(v') > b_{k+1}^\star(v')$,

$$I(b) = \begin{bmatrix} \inf\left\{b_k^\star\left(v'\right) : b_k^\star\left(\tilde{v}'\right) > b_{k+1}^\star\left(\tilde{v}'\right) \forall \tilde{v}' \in \left(v', v\right]\right\}, \\ \sup\left\{b_k^\star\left(v'\right) : b_k^\star\left(\tilde{v}'\right) > b_{k+1}^\star\left(\tilde{v}'\right) \forall \tilde{v}' \in \left[v, v'\right)\right\} \end{bmatrix}.$$

The preceding inequalities and standard sequential arguments imply that the difference between the CDFs H_{m-k+1}^{-i} and H_{m-k}^{-i} is maximized at the interval's right endpoint,

$$\begin{aligned} &H_{m-k+1}^{-i}\left(\min I\left(b_k^\star\left(v\right)\right)\right) - H_{m-k}^{-i}\left(\min I\left(b_k^\star\left(v\right)\right)\right) \\ &\quad < H_{m-k+1}^{-i}\left(\max I\left(b_k^\star\left(v\right)\right)\right) - H_{m-k}^{-i}\left(\max I\left(b_k^\star\left(v\right)\right)\right). \end{aligned}$$

The right endpoint of the interval is either the left endpoint of another interval on which $b_k^\star(\tilde{v}) > b_{k+1}^\star(\tilde{v})$, or of an interval on which $d_+ H_{m-k}^{-i}(b) < d_+ H_{m-k+1}^{-i}(b)$. In the latter case, the difference between the two CDFs is again maximized at the right endpoint of the subsequent interval (and the former case is as analyzed above). In either case, the difference between the CDFs at the right endpoint is increasing in the location of the interval. Since $H_{m-k+1}^{-i} \succeq_{\mathrm{FOSD}} H_{m-k}^{-i}$, it follows that $H_{m-k}^{-i}(\bar{b}) < H_{m-k+1}^{-i}(\bar{b})$ and hence H_{m-k}^{-i} has a mass point at $\bar{b}$. □

Proof of Corollary 4. This is a direct consequence of Lemmas 8 and 9. At $\bar{b}$, $H_{m-k+1}^{-i}(\bar{b}) > H_{m-k}^{-i}(\bar{b})$, contradicting best response behavior. □

C Proofs of equilibrium properties

Proof of Proposition 5. When bids are separating there can be no mass points in the bid distribution. This proof proceeds by ruling out gaps in first-unit bids, then successively ruling out differently-oriented kinks in bids.[39]

Recall from Appendix **??** that separable payoffs for bids for the two units are

$$\begin{aligned} u^1(b; v) &= (v_1 - b_1) H_2(b_1) + \int_0^{b_1} H_1(x)\, dx, \\ u^2(b; v) &= (v_2 - b_2) H_1(b_2) - \int_0^{b_2} H_1(x)\, dx. \end{aligned}$$

[39] In a monotone separating equilibrium, the quantity-monotonicity constraint will never strictly bind. This proof can be adapted to allow for binding monotonicity constraints, but it is not necessary to the point at hand.

The relevant probabilities are

$$
\begin{aligned}
&H_1(x) = F_1(\varphi_1(x))^{n-1}, \\
&H_2(x) = (n-1) F_1(\varphi_1(x))^{n-2} F_2(\varphi_2(x)) (1 - F_1(\varphi_1(x))) + H_1(x).
\end{aligned}
$$

Suppose that there is a gap (discontinuity) in a symmetric equilibrium first-unit bid function b_1. As is clear from the definition of unitwise utility, second-unit bids will never be placed in this gap; H_1 is constant on this gap so unitwise utility is strictly decreasing. Then if there is a gap in the first-unit bid there is a gap in the aggregate market-clearing price distribution. As in other auction models, there is no incentive to bid just above the upper bound of the common gap, implying that this is not possible in an equilibrium without mass points.

Now suppose that there is a downward kink in b_1 at q, so that

$$
\lim_{\varepsilon \searrow 0} \frac{b_1(q) - b_1(q-\varepsilon)}{\varepsilon} > \lim_{\varepsilon \searrow 0} \frac{b_1(q+\varepsilon) - b_1(q)}{\varepsilon}.
$$

Since second-unit bids depend only on the first-unit bid function and there are no mass points, there must be a gap above q in the second-unit bid b_2. Then $F_2 \circ \varphi_2$ is constant just above q, and first-unit bids depend only on $F_1 \circ \varphi_1$. The downward kink in b_1 then implies either a mass point or a gap in b_1 just above q, which we have shown cannot arise.

Now suppose there is an upward kink in b_1 at q, so that

$$
\lim_{\varepsilon \searrow 0} \frac{b_1(q) - b_1(q-\varepsilon)}{\varepsilon} < \lim_{\varepsilon \searrow 0} \frac{b_1(q+\varepsilon) - b_1(q)}{\varepsilon}.
$$

Since the monotonicity constraint is not binding, this implies a gap in b_2 just above q. Per the previous argument regarding the downward kink, this is not possible in equilibrium.

Since there are no kinks in b_1 it is differentiable; this implies (via the same arguments as above) that b_2 is differentiable—non differentiabilities in b_2 will manifest in first-unit incentives, inducing gaps or mass points, which cannot arise. Then b_1 and b_2 are differentiable on their support, and the first-order conditions must be satisfied in a symmetric separating equilibrium. $\square$